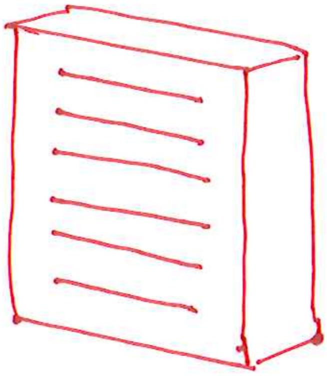
3rd Moscow Workshop on Biomathematics

Institute of Numerical Mathematics RAS



The hybrid discrete-continuous models: asymptotic and numerical study

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OUTLINE

- – Motivation
- – DDA linear version
  - existence/uniqueness
  - homogenization
- – DDA with a feed-back (non-linear setting)
  - existence
  - homogenization
- – Possibility of a partial homogenization
- Conclusion

cell motion

$$\frac{\partial c}{\partial t} = D\Delta c - nc + nG(p)$$

$$\ddot{x} + \mu\dot{x} + H(\nabla n, c) = 0$$

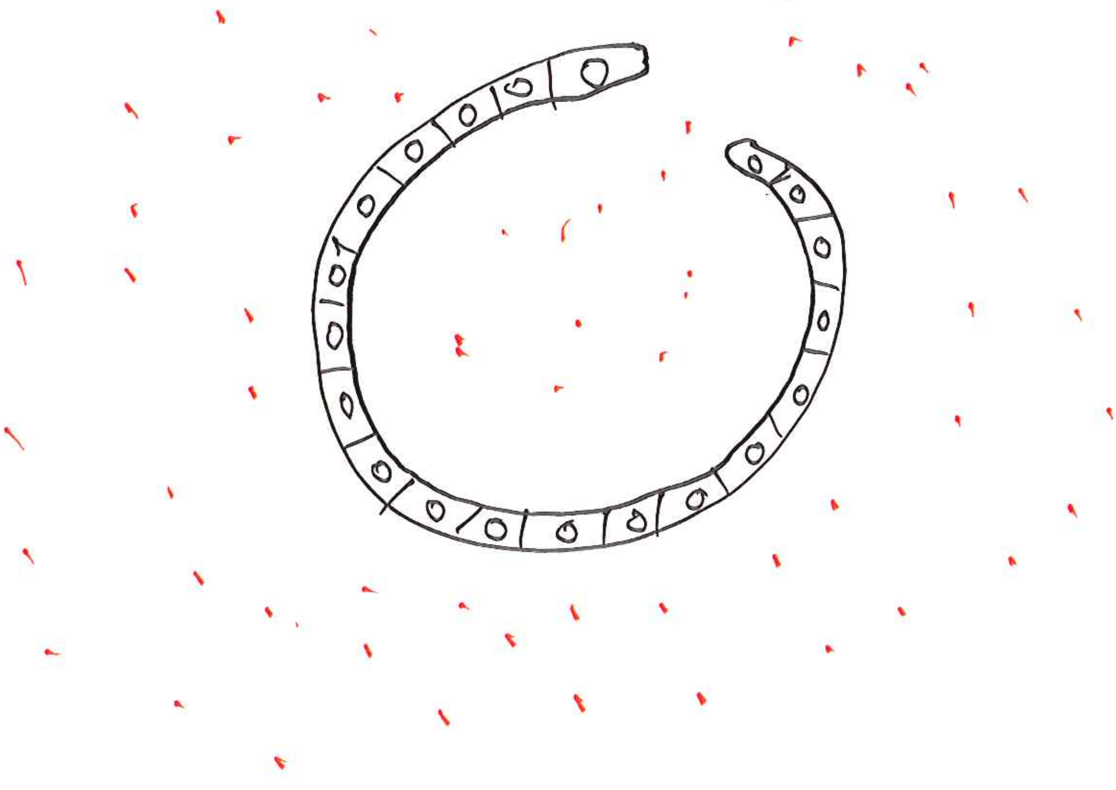
diffusion-absorption

cell division
differentiation
death

$$\frac{dp}{dt} = F(p, c)$$

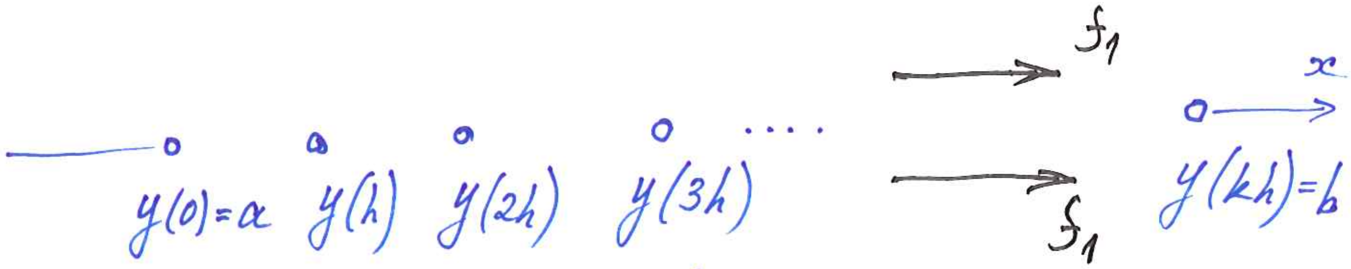
extra-cellular matrix

Anabaena



Motivation

initial position



new coordinates under the force f_1

$$\begin{cases} y''(x) = f_1(x), \ x \in (0,1) \\ y(0) = a, \quad y(1) = b \end{cases} \qquad (1)$$

$-R < a < b < R$,

$f_1 \in C([0,1])$ such that

y increases.

Diffusion-Discrete Absorption

$$\begin{cases} u''(x) - h\sum_{j=0}^{k} \alpha\,\delta(x-y(jh))\,u(x) = f_2(x), \\ \qquad\qquad\qquad x\in(-R,R) \\ u'(-R)=0, \quad u(R)=0 \end{cases} \tag{2}$$

$$-\left(\int_{-R}^{R} u'(x)\varphi'(x)\,dx + \sum_{j=1}^{k} \alpha h\, u(y(jh))\,\varphi(y(jh))\right)$$

$$= \int_{-R}^{R} f_2(x)\,\varphi(x)\,dx \tag{3}$$

$$\forall\, \varphi\in H^1_{0R} = \{\varphi\in H^1(-R,R)\,|\,\varphi(R)=0\}.$$

Theorem 1.1. Let $\alpha, h>0$, $kh=1$, $f_2\in L^2$ $\Rightarrow \exists!$ Sol. of (3)

Homogenization

(continualisation)

$$\begin{cases} \bar{u}''(x) - \alpha \bar{u}(x) \left(y^{-1}(x) \right)' \chi_{[a,b]}(x) = \\ \qquad = f_2(x), \quad x \in (-R, R) \\ \bar{u}'(-R) = 0, \quad \bar{u}(R) = 0 \end{cases} \tag{4}$$

where y is a <u>monotone</u> solution of (1), $f_2 \in C$, $\chi_{[a,b]}$ is the character. func.

Theorem 2.1.

$$\| u - \bar{u} \|_{H^1} = O(h) .$$

$$\|u-\bar{u}\|_{H^1} \leqslant \sqrt{2R^2+1}\,\sqrt{2R}\,Mh,$$

$$M = \alpha\Big(\Big(4\alpha R^{5/2}\frac{1}{K_0}\|f_2\|_{L^2} +$$

$$+\|f_2\|_{L^1}\Big)\Big(|b-a| + 2\|f_1\|_{L^1}\Big) +$$

$$+2R^{3/2}\|f_2\|_{L^2}\Big)$$

$$K_0 = \min_{[0,1]} \int_0^x f_1(\theta)\,d\theta + (b-a) -$$

$$-\int_0^1\int_0^t f_1(\theta)\,d\theta\,dt > 0$$

$$(\,y \nearrow\,).$$

DDA with a feed-back

$$\begin{cases} u''(x) = \alpha h \sum_{j=0}^{K} \delta(x - Y_j^u)\, u(x) + f(x), \\ x \in (-R, R), \\ u'(-R) = 0, \ u(R) = 0, \\ Y_j^u = j h E_0 \left(1 + q F[u]\right)^{-1}, \\ j = 0, \dots, K, \\ F[u] = \int_0^{E_0} u(x)\, dx . \end{cases}$$

$f \in C$, $\alpha > 0$, $h > 0$, $Kh = 1$;

$0 < E_0 < R/2$, $q > 0$.

<u>Theorem 3.1</u> $\exists\ \alpha_*, q_* > 0$, indep. of h, such that if $0 < \alpha < \alpha_*$, $0 < q < q_*$ then $\exists$ a solution.

Proof:

Schauder fixed point $\boxed{Tu = u}$ theorem for

$$T : C([-R,R]) \to C([-R,R])$$

$$(Tv)(x) = -\int_x^R \left\{ \Lambda v(t) + \int_{-R}^t f_2(\theta)\,d\theta \right\} dt$$

$$(\Lambda v)(x) = h\alpha \sum_{j : Y_j^v \le x} v(Y_j^v)$$

Homogenization (continualisation)

$(y^{-1})' = \mathrm{const}$

$$
\begin{cases}
\bar{u}''(x) = \alpha \dfrac{1+qF[u]}{E_0} \chi_{[0, Y_K^u]}(x)\,\bar{u}(x) + f(x), \quad x \in (-R, R), \\
\bar{u}'(-R) = 0, \quad \bar{u}(R) = 0
\end{cases}
$$

Theorem 4.1. $\underbrace{\exists\, \alpha_*, q_* > 0}_{\text{indep. of } h}$, such that if $0 < \alpha < \alpha_*$, $0 < q < q_*$, then $\exists!$ solution in a ball.

Theorem 4.2. For

$0 < \alpha < \alpha_*$, $0 < q < q_*$,

$$\| u - \overline{u} \|_C = O(h)$$

Partial homogenization

$$\begin{cases} \bar{u}''(x) - \alpha \bar{u}(x) \left(y^{-1}(x)\right)' \chi_{[a,b]}(x) = \\ \qquad = f_2(x), \quad x \in (-R, R) \setminus [c,d] \\ \bar{u}''(x) - h \sum\limits_{j:\ y(jh) \in [c,d]} \alpha \delta(x - y(jh)) \bar{u}(x) \\ \qquad = f_2(x), \quad x \in \underline{\underline{[c,d]}} \\ [\bar{u}] = 0, \quad [\bar{u}'] = 0 \quad \text{in } c, d, \\ \bar{u}'(-R) = 0, \quad \bar{u}(R) = 0 \end{cases}$$

It is a hybrid model

<u>Theorem 5.1.</u> $\exists!$ sol.

$$-\Big(\int_{-R}^{R} \bar{u}'(x)\,\varphi'(x)\,dx \; +$$

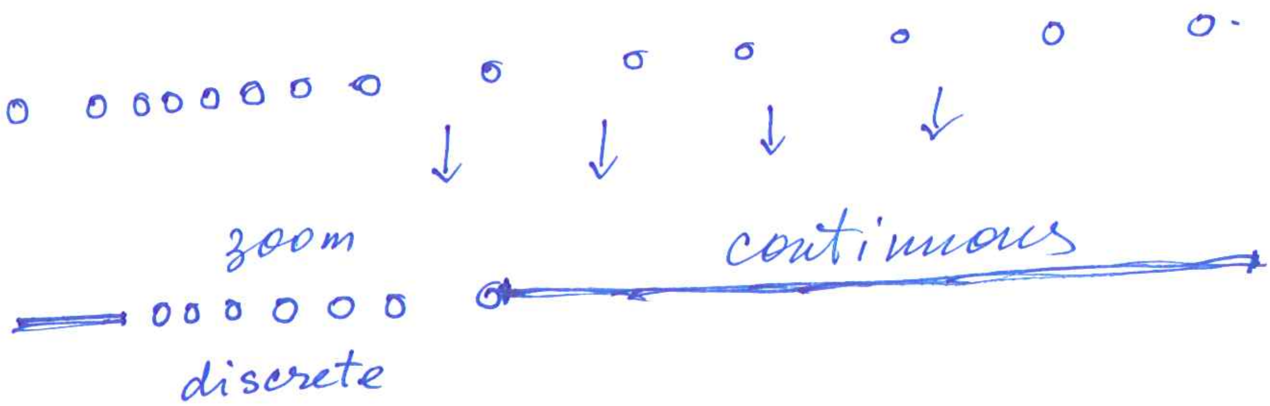
$$+ \int_{[a,b]\setminus[c,d]} \alpha\, \bar{u}(x)\,\varphi(x)\,\big(y^{-1}(x)\big)'\,dx \; +$$

$$+ \sum_{j:\, y(jh)\in[c,d]} \alpha h\, \bar{u}\big(y(jh)\big)\,\varphi\big(y(jh)\big) =$$

$$= \int_{-R}^{R} f_2(x)\,\varphi(x)\,dx$$

$$\forall\, \varphi \in H^1_{0R}$$

Theorem 5.2. $\|u-\bar{u}\|_{H^1} = O(h)$.



Conclusion

Passage micro → macro
in discrete models DDA

Possibility of combined
hybrid description with
the discrete zoom.

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GRANTS

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