pp. 1–10 (2019)

Stability indicatrices of nonnegative matrices and some of their applications in problems of biology and epidemiology

V. N. Razzhevaikin*

Abstract — The method of constructing a stability indicatrix of a nonnegative matrix having the form of a polynomial of its coefficients is presented. The algorithm of construction and conditions of its applicability are specified. The applicability of the algorithm is illustrated on examples of constructing the stability indicatrix for a series of functions widely used in simulation of the dynamics of discrete biological communities, for solving evolutionary optimality problems arising in biological problems of evolutionary selection, for identification of the conditions of the pandemic in a distributed host population.

Keywords: Nonnegative matrix, stability indicatrix, evolutionary optimality, emergence of a pandemic

MSC 2010: 92D15, 15A42, 92D30

The most common structuring in problems of mathematical modelling of the dynamics of biological communities of species is the structuring by age. For example, for a linear discrete model the dynamics of the population with age structure is described by the Leslie model (see [4]), its exponential dynamics is determined by properties of a nonnegative matrix (Leslie matrix) linking together the states of the population at two consecutive steps in time. In this case the presence of growth or decrease depends solely on whether the spectral radius of this matrix is greater than or less than one. It turns out that to answer this question it is sufficient to calculate some value specified very simply and called the *stability indicatrix* of the matrix (potential growth indicator in the terminology of [6]). For the Leslie matrix this value can be constructed on the basis of a suitable representation of its characteristic polynomial. At the same time it has a meaningful biological interpretation as the biological potential, i.e., the average number of births produced throughout the life of one individual. An important feature of the biological potential is the ability to serve as a selection functional in models of evolutionary optimality (see [7] for formulation of problems of this kind).

The ability to construct the biological potential on the basis of the characteristic polynomial of the original matrix only is also admitted for a number of generalizations of the Leslie matrix, these are the so-called Lefcovitch matrices [3] and the Logofet matrix most general in this series [5]. In the models given by these matrices the population is structured according to some ordered set of life stages passed by

*Federal Research Center 'Informatics and Control' of the Russian Academy of Sciences, Moscow 119333, Russia. E-mail: razzh@mail.ru

each individual in the direction of this order.

The aim of this paper is to study the possibility of constructing the stability indicatrix for an arbitrary nonnegative matrix in the context of the use of this indicatrix to realize the above-mentioned functions of biological potential.

It should be noted that if we restrict ourselves to stability issues only, then we can use a large set of criteria establishing it, including the criteria of nondegeneracy of M-matrices (see, e.g., [2]). Many of them are quite suitable for constructing stability indicatrices in the roles indicated here. However, there are such among them as, for example, the positivity of leading principal minors, which in degenerate cases can produce a function not suitable for the role of an indicatrix. The direction we have chosen is aimed at construction of the most simple formulas from the coefficients of the matrix convenient for operating with them in their general form, which allows us, as it seems, not to overload the presentation with well-known results.

Now we describe briefly the content of the paper. Section 1 presents the formulation of the main result in the form of a theorem on ability to construct a polynomial indicatrix for a nonnegative matrix. This is supplied with a brief draft of its proof and detailed description of the algorithm and its application. The remaining sections are mainly illustrative. Section 2 presents an example of the use of the stability indicatrix in the case of a matrix participating in the classic population biology model of Logofet [5]. We consider its particular cases for the Lefcovitch and Leslie matrices mentioned above. Section 3 presents one variant of discrete problems of evolutionary optimality where stability indicatrices can be used for selection functionals. Section 4 presents the classical model of the spread of epidemic in the conditions of distributed population of the host. The problem of emergence of the pandemic and conditions associated with it can be solved in this case by constructing stability indicatrices of suitable nonnegative matrices.

1. The main result

Let Ω be an open set in $\mathbf{R}^n = \{x = (x_1, \ldots, x_n)\}$, $x_i \in \mathbf{R}$. We say that a certain property is fulfilled *zero-strongly almost everywhere* on Ω if for any subset $M \subset \{1, \ldots, n\}$ of cardinality $|M| \geqslant 1$ it is fulfilled on some open and everywhere dense in $\Omega_M = \Omega \cap \left\{x : \sum_{i \notin M} x_i^2 = 0\right\}$ set $\Omega'_M \subset \Omega_M$ for which the Lebesgue measure of dimension $|M|$ is reduced to zero, i.e., $\mathrm{mes}_{\mathbf{R}^{|M|}}(\Omega_M \backslash \Omega_M{}') = 0$.

A function $F(x)$, $F : \Omega \to \mathbf{R}$, is called *polynomially specified* on the set Ω if there exist $N(n) \in \mathbf{N}$ and a mapping $P^F : \Omega \to \mathbf{R}^{N(n)}$, $P^F(x) = \left(P_1^F(x), \cdots, P_{N(n)}^F(x)\right)$, such that

$$F(x) = \max_{j=1,\ldots,N(n)} P_j^F(x). \tag{1.1}$$

Here $P_j^F(x)$ are polynomials of n variables $x_l \in \mathbf{R}$, $l = 1, \ldots, n$, such that $x = (x_1, \ldots, x_n) \in \Omega$ with coefficients from $\mathbf{R}$ dependent on the function F only. The minimal value $N(n)$ is called the *size* of polynomial specification of the function $F(x)$.

Let $r(A)$ be the spectral radius (or Perron root) of a nonnegative $n \times n$ matrix $A = (a_{ij})$. The *stability indicatrix* of this matrix on the set $\Omega \subset \mathbf{R}_+^{n^2} = \{a_{ij} \geqslant 0;\ i,j = 1,\dots,n\}$ is said to be the function $\Phi(A)$ of its coefficients $\{a_{ij}\}$, $\Phi : \Omega \to \mathbf{R}$, such that the sign of $(\Phi(A) - 1)$ coincides with the sign of $(r(A) - 1)$. In particular, the Perron root of the matrix is such an indicatrix. It is also evident that each strictly monotone increasing function of the indicatrix taking the value one at the argument one may serve as an indicatrix.

A polynomially specified indicatrix of a nonnegative matrix is called its *polynomial* stability indicatrix.

Theorem 1.1. *The polynomial stability indicatrix of size n of a nonnegative $n \times n$ matrix zero-strongly exists almost everywhere for any $n \in \mathbf{N}$ on the set $\mathbf{R}_+^{n^2}$ of coefficients of that matrix.*

The idea of the proof (see [9]) consists in inductive reduction of the dimension of the matrix from which the stability indicatrix is constructed. If $A_k = \left(a_{ij}^k\right)$ is a $k \times k$ matrix and $A_n = A$, then, applying the indicative recalculations by the formulas

$$a_{ij}^{k-1} = a_{ij}^k + \frac{a_{ik}^k a_{kj}^k}{1 - a_{kk}^k}, \quad i,j = 1,\dots,k-1 \tag{1.2}$$

in the case of positive denominator we can construct for the known matrix $A_k \geqslant 0$ a new matrix $A_{k-1} \geqslant 0$ having the same location of the spectral radius relative to the unit as for the matrix A_k.

Initially, for the indicatrix of stability we consider the value of the retrospective solution to a discrete boundary value problem for a nonnegative n-dimensional vector in a strip infinite in inverse time, unit values are posed at all moments except for the last one on the border of that strip. In this case the transition to the next moment for non-fixed components of the vector is performed by a linear transformation with the use of the original matrix. It occurs that in the case of its indecomposability the last value for the selected component can serve as a stability indicatrix of the original matrix if such solution exists. This can be easily seen if we interpret the indicated transitions as a flow of some substance distributed over n cells with the matrix coefficients as coefficients of transition. Truncation or supplement to the unit value occurs in this case on the boundary of the strip. The actual action is determined by the last boundary value. Since it is obtained by summation of the flows over all possible paths, we get an opportunity to obtain it by successive narrowing the strip. Each next its contraction is equivalent to a loss of paths including transitions between the remaining cells (ith and jth ones) and one removed cell (kth). The delay on the removed cell may take an arbitrary number (say, m) of iterations each of which changes the quantity of the substance in the kth cell by a_{kk}^k times. This explains the denominator in (1.2) determined by the sum of the series $\sum_{m=0}^{\infty} (a_{kk}^k)^m$. The numerator is responsible for the indicated transitions $j \to k$ and $k \to i$. The result for the indecomposable case can be extended to the case of decomposable matrices taking into account the absence of changes in blocks of the matrix corresponding to cells not cyclically connected with the removed cell due to the graph of the matrix.

If we get the matrix $A_1 = \left(a_{11}^1\right)$, then the value a_{11}^1 may serve as the stability indicatrix of the matrix A. It has the form of a multilevel fraction of its coefficients. Applying to it the procedure of reducing the levels based on replacement of the fraction $\gamma_l = \alpha_{l-1}/\beta_{l-1}$ by the fraction $\gamma_{l-1} = \alpha_{l-1} - \beta_{l-1} + 1$, we can finally obtain the indicatrix in the form of a polynomial of the coefficients of the matrix A.

The first occurrence of a negative denominator at the lth step ($l = n-k+1$) of application of procedure (1.2) is equivalent to the inequality $a_{kk}^k > 1$ implying the inequality $r(A) > 1$. In this case the value a_{kk}^k also has the form of a multilevel fraction of the coefficients of the matrix A, and the polynomial of those coefficients can be also constructed according to the same scheme to compare it with the unit. The set of n polynomials constructed for all a_{kk}^k, $k = 1,\dots,n$, can be used in the construction of a polynomially specified stability indicatrix of the matrix A in accordance with (1.1).

The degenerate cases $a_{kk}^k = 1$, $k = 2,\dots,n$, correspond to sets of coefficients of the matrix A which are excluded by the conditions of the theorem.

For problems of pure computational nature we have no need to construct indicatrices in form of polynomials. In this case the recalculation by formulas (1.2) gives the required result if we use the following scheme.

1. The elements from $\{1,\dots,n\}$ are ordered arbitrarily (below we keep their usual order).

2. Specify $A_n = A$ with $a_{ij}^n = a_{ij}$ and $J_{n+1} = 0$.

3. Apply induction over k decreasing from n to 2.

3.1. Specify $J_k = \max\{J_{k+1}, a_{kk}^k\}$.

3.2. In the case $J_k > 1$ we conclude that $r(A) > 1$.

3.3. In the case $J_k \leqslant 1$ we calculate the elements a_{ij}^{k-1}, $i, j \in \{1,\dots,k-1\}$, of the new nonnegative matrix A_{k-1} by formulas (1.2).

4. In the case $J_k \leqslant 1$ for all $k = n, n-1,\dots,2$ the value $J_1 = \max\{J_2, a_{11}^1\}$ is taken as the stability indicatrix of the nonnegative matrix A.

Remark 1.1. If event 3.2 has occurred, we have no need to continue the calculations. If we forcibly continue calculations in accordance with case 3.3 without the restriction indicated there, then the conclusion of case 4 remains valid independently of their result without the restriction indicated in that case.

Remark 1.2. The case $a_{kk}^k = 1$ results in the zero denominator in (1.2). To include it into the common scheme, it is convenient to assume that J_k takes values on $\mathbf{R} \cup +\infty$ nominally assuming that $\infty \times 0 = 0/0 = 0$. Other possible uncertainties can take arbitrary values. The validity of these conventions is easily seen within the above flow interpretation. Using them, we can remove the restrictions indicated in the formulation of the theorem in its part related to the construction of the indicatrix in the form of a polynomial for its variant in the form of a multilevel fraction.

Remark 1.3. The detailed analysis shows that, changing the enumeration of rows and columns of the matrix A to the opposite one, the expressions $(1-a_{kk}^{k})$ in (1.2) tested for positivity coincide with the diagonal elements of the LU-decomposition of the matrix $(I-A)$ (see, e.g., [10]). Note that such positivity is sufficient for this matrix to be a nondegenerate M-matrix.

Remark 1.4. A large number of stability criteria for nonnegative matrices (see, e.g., [2]) can create an impression of triviality of solution of stability indicatrix problems by expansion of the role of functions used in these criteria to the role of indicatrices. Here we need an accurate verification of the degenerate cases. For example, the fact that the leading minors of the matrix $(I-A)$ turn to zero for $A=\operatorname{diag}\{1,2\}$ does not mean that the spectral radius of the matrix A equals one.

2. Some examples

Logofet matrix (see [5]) of dimension n is given by its canonical (i.e., after appropriate simultaneous renumbering of rows and columns) form as

$$A_{\mathrm{Lg}}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,n-1} & a_{1,n} \\ a_{21} & a_{22} & \dots & 0 & 0 \\ a_{31} & a_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

with $a_{ij}\geqslant 0$ (so that $a_{ij}=0$ for $j>i>1$).

A meaningful interpretation of the population dynamics model with such a matrix can be represented as a motion of each individual specimen over stages of development in one direction (i.e., the stages are ordered and the transition from a later stage to an earlier one is prohibited). The transition is not necessarily sequential and can admit jumps over several stages at once (see about this in [6]). A partial return to the initial stage from later ones is interpreted as the birth of new individuals by parents being at those later stages.

The calculation of the stability indicatrix for the Logofet matrix can be performed based on its characteristic polynomial. It has the form

$$P(A_{\mathrm{Lg}},\lambda)=\prod_{j=1}^{n}(\lambda-a_{jj})\cdot\big(1-Q(A_{\mathrm{Lg}},\lambda)\big) \tag{2.1}$$

with

$$Q(A_{\mathrm{Lg}},\lambda)=\sum_{i=2}^{n}a_{1i}\frac{\sum\limits_{m_i}\prod\limits_{l=1}^{k_{m_i}}a_{p_l^{m_i},p_{l-1}^{m_i}}}{\prod\limits_{j=1}^{i}(\lambda-a_{jj})} \tag{2.2}$$

where the numerator in (2.2) contains under the sign of sum the products of all ordered (so that $p_l^{m_i} > p_{l-1}^{m_i}$) subsets $m_i = \left\{p_l^{m_i}\right\}$ of the set $\{1,\dots,i\}$ with the beginning equal to one (so that $p_1^{m_i} = 1$) and the maximal value equal to i (so that $p_{k_{m_i}}^{m_i} = i$). In this case we nominally assume $a_{p_1^{m_i},p_0^{m_i}} = 1$ for any m_i.

For $a_{jj} < 1$ the function $\Phi(A_{\rm Lg}) = Q(A_{\rm Lg},1)$ may serve as a stability indicatrix because in this case the function $Q(A_{\rm Lg},\lambda)$ decreases strictly and monotone in λ for $\lambda > \max a_{jj}$ and the first summand in the right-hand side of (2.1) is positive. As was shown in [8], the Logofet matrix is the 'maximal' nonnegative matrix up to simultaneous renumbering of rows and columns which admits in these conditions a stability indicatrix expressed by its characteristic polynomial. This maximality is understood in the sense of the number of admissible nonzero elements.

It is not difficult to check that in the case of the order providing the canonical form of the Logofet matrix the successive application of the recalculation by formulas (1.2) keeps all elements of the matrix except for the elements of the first row. Moreover, the recalculation by these formulas leads to the relation $a_{11}^1 = \Phi(A_{\rm Lg})$, i.e., the algorithm proposed here leads to formulas obtained from characteristic polynomials. In particular, using this algorithm, one can obtain classic formulas for stability indicatrices of some particular cases of the Logofet matrix.

Leslie matrix $A_{\rm Les}$ (see [4]) is obtained from the Logofet matrix under the assumption that only the age-specific fertility rates $b_i = a_{1i}$ and rates of survival $s_i = a_{i+1,i} \in (0,1]$ for an individual of age i to live to the age $i+1$ are distinct from zero. In this case the stage is the age; the stability indicatrix coincides with the biological potential of the population $\Phi(A_{\rm Les}) = \sum_{i=1}^n b_i \prod_{j=1}^i s_{j-1}$.

Lefcovitch matrix $A_{\rm Lf}$ (see [3]). In this case an individual passes the stages sequentially but, in contrast to the Leslie model, can stay in those stages. In addition to nonzero coefficients of the Leslie matrix, the coefficients of delay at the ith stage $r_i \in [0,1)$ can be positive in this case. The biological potential equals $\Phi(A_{\rm Lf}) = \sum_{i=1}^n b_i \prod_{j=1}^i s_{j-1}/(1-r_j)$ here.

Note that, due to the indicated above property of preservation of elements positioned not on the first row in calculation by formulas (1.2), the form of both these matrices is also preserved.

3. Evolutionary optimality in distributed discrete systems

We consider a biological community with the dynamics described by the following discrete dynamical system:

$$\begin{cases} x^{m+1} = A(x^m,y^m)x^m \\ y^{m+1} = B(x^m,y^m). \end{cases} \tag{3.1}$$

Here $m \in \mathbf{N}$ is the discrete time; $x^m = (w_1^m,\dots,w_L^m)$ is the community vector composed of the collection of population size vectors $w_l^m = (u_{l,1}^m,\dots,u_{l,n_l}^m)$ structured according to a certain discrete set of indicators (those may be the age stage of life of population individuals, habitat, etc.); n_l is the number of indicators in the lth type of community, $l \in \{1,\dots,L\}$); $y^m = (y_1^m,\dots,y_Y^m)$ is the vector of external factors; A is a block-diagonal (in blocks of vectors w_l^m) nonnegative $N\times N$-matrix of state changes in the community under passing from the time moment m to the moment $m+1$; $N=\sum_{l=1}^{L} n_l$; B is a Y-vector.

In model (3.1) we also assume that the elements of the matrix $A(x,y)$ whose row and column relate to different l are equal to zero. This means that in the process described by the dynamics considered here the species are not transformed to each other, but reproduce only their own kind. In this case the matrix $A(x,y)$ is split into diagonal blocks $A^l(x,y)$ so that each its block determines the character and rates of transitions for a particular population subject to possible birth and death processes. The rates of all those processes depend on the state of the community as a whole including the current set of external factors. The dynamics specification rules for the latter ones can be quite arbitrary.

Let us suppose that system (3.1) has a stable equilibrium position $(\bar{x},\bar{y})$ with $\bar{x} = (0,\dots,0,\bar{w}_{K+1},\dots,\bar{w}_L)$ and $\bar{w}_l > 0$ (component-wise inequality) for $l = K+1,\dots,L$ and $0 < K < L$. This means that the spectrum of the Jacobian $J = J(\bar{x},\bar{y})$ of the right-hand side of the system calculated at this equilibrium position is localized in the unit circle of the complex plane. Assuming the block $(K, L-K, Y)$ form in the components (ζ,ξ,y) with $\zeta = (w_1,\dots,w_K)$, $\xi = (w_{K+1},\dots,w_L)$, this Jacobian takes the form

$$J = \begin{pmatrix} A^{\zeta}(\bar{x},\bar{y}) & \mathbf{0} & \mathbf{0} \\ A^{\xi}_{\zeta}(\bar{x},\bar{y})\cdot\bar{\xi} & A^{\xi}_{\xi}(\bar{x},\bar{y})\cdot\bar{\xi} + A^{\xi}(\bar{x},\bar{y}) & A^{\xi}_{y}(\bar{x},\bar{y})\cdot\bar{\xi} \\ B_{\zeta}(\bar{x},\bar{y}) & B_{\xi}(\bar{x},\bar{y}) & B_{y}(\bar{x},\bar{y}) \end{pmatrix}.$$

Here $\bar{\xi} = (\bar{w}_{K+1},\dots,\bar{w}_L)$, $A^{\xi}(\bar{x},\bar{y})$ is the restriction of the operator (corresponding diagonal block of the matrix) A onto the components ξ (and similarly for ζ), the subscript means the calculation of the corresponding Jacobian of the mapping into the space of matrices at the point $(\bar{x},\bar{y})$, the dot after the derivative means the place of substitution of the corresponding component of the vector under the action of the matrix as an operator (the absence of dot means the substitution from the right).

The triangle form of J and the assumed stability imply that the following inequality for the spectral radius holds for any $l \leqslant K$:

$$r(A^l(\bar{x},\bar{y})) \leqslant 1. \tag{3.2}$$

On the other hand, since $\bar{w}_l = A^{l'}(\bar{x},\bar{y})\bar{w}_l$ is positive, for $l' > K$ we have $r(A^{l'}(\bar{x},\bar{y})) \geqslant 1$, and for the matrices $A^{l'}(\bar{x},\bar{y})$ with nonnegative components we have the equality because for nonnegative matrices only a Perron eigenvalue can have a positive eigenvector.

Taking into account (3.2), for such matrices we obtain the relation

$$1 = r(A^{l'}(\bar{x},\bar{y})) = \max r(A^l(\bar{x},\bar{y})) \tag{3.3}$$

where l can run over all possible L values and l' runs over last $L-K$ ones corresponding to nonzero components of the vector $\bar{x}$. The second equality in (3.3) is called the principle of *evolutionary optimality* (see, e.g., [7]), which allows us to calculate the characteristics of species survived in the equilibrium in the solution of the extreme problem in (3.3).

The solution of problem (3.3) is often far from simple and in any case is not formalized analytically in the general formulation. A noticeable advance in its study with the aim to use it for analytical constructions is possible with the availability to replace the function optimized in (3.3) by a simpler one. Such a possibility appears in the case of replacement of the spectral radius $r(A^l(\bar{x},\bar{y}))$ by the stability indicatrix of the matrix $A^l(\bar{x},\bar{y})$ calculated in the stable equilibrium position $(\bar{x},\bar{y})$.

If $\Phi(A^l(\bar{x},\bar{y}))$ is such indicatrix, then problem (3.3) can be replaced by the problem

$$\Phi(A^{l'}(\bar{x},\bar{y})) = \max \Phi(A^l(\bar{x},\bar{y})). \tag{3.4}$$

If we use polynomial indicatrices, problem (3.4) becomes essentially simpler than problem (3.3). In the rough case we do not need to take into account all polynomials in formula (1.1) except for the last one corresponding to a_{11}^1. This relates to the assumption of the stability of the equilibrium state excluding in this case the expansion of the spectral radius of the matrices A_k obtained in intermediate stages of iterative process (1.2) out of the inner part of the unit circle. As the result, the calculation of the selection function in (3.4) requires only elementary operations. The situation when the stable equilibrium state of the community has been formed so that the information about extinct species was lost the solution of problem (3.4) allows us to obtain values of evolutionary significant (i.e., those used in the selection) parameters for the steady state distribution and those parameter can take values from an infinite-dimensional space (for example, they can be species-specific functions of intra- and inter-population interaction).

4. Model of the epidemic in the presence of migration

Suppose the population is distributed over several regions numbered by the index $i = 1,\dots,n$, while N_i is the total population of the ith region, $I_i(t)$ is the number of infected individuals in it at the time moment t, $S_i(t)$ is the number of susceptible ones, $R_i(t) = N_i - I_i(t) - S_i(t)$ the number of removed ones.

It is assumed that the residents of each region spend a part of their time outside sharing it for other regions. By $\varepsilon_{ij} \geqslant 0$ we denote the part of time spent by the residents of the ith region in the jth region so that ε_{ii} is the time spent at home and $\sum_{j=1}^n \varepsilon_{ij} = 1$.

Under these assumptions the dynamics of the epidemic is described by the following system of $2n$ equations generalizing the classic Kermack–McKendrick model [1]):

$$\begin{cases} \dfrac{\mathrm{d}S_i}{\mathrm{d}t} = -\alpha S_i \left(\sum\limits_{k=1}^{n} \varepsilon_{ik} \sum\limits_{j=1}^{n} \varepsilon_{jk} I_j \right) + \gamma (N_i - S_i), \quad i = 1, \dots, n \\ \dfrac{\mathrm{d}I_i}{\mathrm{d}t} = \alpha S_i \left(\sum\limits_{k=1}^{n} \varepsilon_{ik} \sum\limits_{j=1}^{n} \varepsilon_{jk} I_j \right) - (\beta + \gamma) I_i, \quad i = 1, \dots, n \end{cases} \tag{4.1}$$

where $\alpha > 0$ is the coefficient of contagion defined as the probability for the susceptible individual to get infected within the unit of time from a contact with an infected person, $\gamma > 0$ is the renewal rate equal to the natural mortality rate (it is calculated as the inverse of life expectancy), $\beta > 0$ is the coefficient of recovery (inversely proportional to the duration of the disease in its active, i.e., contagious phase).

System (4.1) implies the following equation for the dynamics of removed ones

$$\frac{\mathrm{d}R_i}{\mathrm{d}t} = \beta I_i - \gamma R_i$$

and the invariance of the domain

$$U = \{S_i \geqslant 0, I_i \geqslant 0, S_i + I_i \leqslant N_i, i = 1, \dots, n\}.$$

The probability of a pandemic is associated with the instability of the 'trivial' equilibrium position $\{S_i^* = N_i, I_i^* = 0, i = 1, \dots, n\}$.

The case of its stability corresponds to the localization of all $2n$ (taking into account the multiplicity) eigenvalues of the Jacobian J of the right-hand side of system (4.1) calculated at this equilibrium position in the left complex half-plane. It has the form

$$J = \begin{pmatrix} -\gamma \mathbf{1} & -\alpha \operatorname{diag}(\mathbf{N}) \mathrm{E}\mathrm{E}^{\mathrm{T}} \\ \mathbf{0} & \alpha \operatorname{diag}(\mathbf{N}) \mathrm{E}\mathrm{E}^{\mathrm{T}} - (\beta + \gamma) \mathbf{1} \end{pmatrix} \tag{4.2}$$

where the cells correspond to $n \times n$ blocks, $\mathbf{0}$ is the zero and $\mathbf{1}$ is the identity $n \times n$ matrices, $\mathbf{N} = (N_1, \dots, N_n)^T$, $\operatorname{diag}(\mathbf{N})$ is the diagonal matrix with the vector $\mathbf{N}$ on the main diagonal, $\mathrm{E} = (\varepsilon_{ij})$ is the $n \times n$ matrix of correspondence.

Since the Jacobian J has a block-triangular form, its spectrum coincides with the union of spectra of its diagonal blocks. The first of them is diagonal with the value $-\gamma < 0$ so that the stability of the equilibrium position $\{S_i^*, I_i^*, i = 1, \dots, n\}$ corresponds to localization of eigenvalues of the second diagonal block in (4.2) in the left complex half-plane. It is equivalent to the localization of the spectrum of the nonnegative $n \times n$ matrix $A' = \alpha/(\beta + \gamma) \operatorname{diag}(\mathbf{N}) \mathrm{E}\mathrm{E}^T$ with the coefficients $a'_{ij} = \alpha N_i/(\beta + \gamma) \sum_{k=1}^{n} \varepsilon_{ik} \varepsilon_{jk}$ inside the unit circle, i.e., to the inequality $r(A') < 1$.

The use of the indicatrix of the matrix A' constructed in accordance with the scheme presented above instead of the spectral radius solves the problem of emergence of a pandemic within this model.

References

1. N. T. J. Bailey, *The Mathematical Approach to Biology and Medicine.* Wiley, London, 1967.
2. A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences.* SIAM, Philadelphia, 1994.
3. L. P. Lefcovitch, The study of population growth in organisms grouped by stages. *Biometrics* (1965) **21**, 1–18.
4. P. H. Leslie, On the use of matrices in certain population mathematics. *Biometrica* (1945) **33**, 183–212.
5. D. O. Logofet, Convexity in projection matrices: projection to a calibration problem. *Ecological Modelling* (2008) **216**, No. 2, 217–228.
6. D. O. Logofet and I. N. Belova, Nonnegative matrices as an instrument for modelling population dynamics: classic models and modern generalizations. *Fundam. Prikl. Matem.* (2007) **13**, No. 4, 145–164 (in Russian).
7. V. N. Razzhevaikin, The use of evolutionary optimality principles in simulation of structured biosystems. *Russ. J. Numer. Anal. Math. Modelling* (2014) **29**, No. 5, 297–305.
8. V. N. Razzhevaikin, Nonnegative matrices admitting an elementary stability indicatrix. In: *Modelling, Decomposition and Optimization of Complex Dynamic Processes*. Vol. 31, No. 1. Comp. Centre RAS, Moscow, 2016. pp. 138–154 (in Russian).
9. V. N. Razzhevaikin, On the ability to construct a polynomial stability indicatrix of nonnegative matrices. *Operations Research (Models, Systems, Solutions)* (2018) **4**, No. 13, 3–12.
10. D. S. Watkins, *Fundamentals of Matrix Computations*. Wiley, New York, 2010.