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From chaos to order. Difference equations in one ecological problem

G. K. Kamenev[*], O. P. Lyulyakin[*], D. A. Sarancha[*], N. A. Lysenko[†], and V. O. Polyanovsky[‡]

Abstract — In this paper we consider properties of the difference equations (discrete mappings) obtained in the study of the population dynamics of lemmings. A bifurcation scenario is proposed for obtained equations. Certain stability zones appear under this scenario with periods varying in order of natural series and also zones with more complicated modes. The study of transitional zones ('ordering of the chaos') is performed with the use of analytic calculations and computational experiments. Numerical analysis of mappings uses the methods of approximation of implicitly specified sets allowing us to construct and visualize sets of 'resonance' parameters including the front of the so-called singularity of 'blue sky'.

Keywords: Difference equations, discrete mappings, computational experiment, methods of approximation of implicitly specified sets, interactive decision maps.

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Since the end of last century the interest in difference equations (DE) was largely driven by environmental problems. The initial impetus was given by R. May in [8] where the term 'chaos' was used in model description of biological populations by a logistic equation, this term was introduced by T. Li and J. York [5], 'the cycle of period three creates a chaos'. More precisely, as was shown earlier by A. N. Sharkovsky [12], the existence of a cycle of period three implies the existence of a cycle of any period (in the continuous mapping of a unit segment onto itself). In fact, with the filing of R. May [8] the logistic equation was the focus of studying the difference equations [2].

Describing the population dynamics of animals within the framework of mathematical models of tundra populations and communities [1, 11, 13], we succeeded in justification of the type of difference equation (DE) differing from conventional logistic equation. It also represents a unimodal mapping of the unit segment onto itself and consists of three line segments, two of which have the absolute value of the derivative exceeding one, and the third is a constant (a segment of horizontal

[*]Dorodnicyn Computing Centre of the RAS, Moscow 119333, Russia

[†]Moscow Aviation Institute (National Research University), Moscow 125993, Russia

[‡]Engelhardt Institute of Molecular Biology of the RAS, Moscow 119991, Russia. E-mail: polyanovskyvo@yandex.ru

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line—‘plateu’), the numerical value of this constant is taken as the bifurcation parameter.

For this type of equations and the chosen bifurcation parameters, a scenario of its change was proposed so that stability zones with stable cycles appear sequentially [9]. Inside the zone of stability the period of cycles is constant, in the transition from one zone to another the period changes according to the sequence of natural numbers. Stability zones are separated from each other by transitional domains with more complex (quasistationary) modes. In this case the domains of stability (for parameters of the equation allowing one to reproduce the population dynamics close to the real one) are much wider than the transitional zones. In contrast with traditional studies [2], the emphasis shifted here, the purpose of this research is not just to prove the existence of cycles of different periods, but the search for a region of their stability. The representation of DE in the form of straight line segments makes the problem of determination of periodic orbits and stability domains solvable by analytic methods. This allows one to analyze the so-called region of chaos (where trajectories are very sensitive to small changes of parameters) by using the procedure of consecutive consideration of cycles with increasing periods, i.e., to ‘order the chaos’ or indicate the sequence of the period appearance in the considered domain for increasing length of periods.

In most cases the period of oscillation of trajectories is not sensitive to small changes (with the exception of boundary points) in contrast to transition zones where the periods of cycles are very sensitive to small changes of the bifurcation parameter.

After appearance of the period three (as shown in this example) the stall in the chaos does not occur, but we have its own laws and stability domains of cycles (with periods greater than or equal to four). This paper is focused on revealing these features.

Origin of the problem. In the quantitative analysis of ecological processes the first problem is the choice of equations for model description. The ecological science has no prevalent equations. Analysis of particular processes leads to refinement and sometimes to a fundamental review of used equations.

In our research the problem of studying properties of difference equations (DE) appeared in the description of the population dynamics of animals with mathematical models of tundra populations and communities [1, 11, 13]. The studies resulted in the construction of a set of interconnected models. The base of such set is formed by detailed imitation models constructed in cooperation with biologists on the base of dependencies approved by experts and taking into account season variations of parameters. The analysis of results of numerical experiments with mutually complementary models of the ‘vegetationlemmingsArctic foxes’ community and the population of lemmings subject to age structure had led to justification of a simplified model in the form of a difference equation connecting the population size of lemmings in two subsequent years. Using the obtained equation [11, 13], we succeeded in reconstruction of time evolution dynamics qualitatively close to the actual population dynamics of lemmings [10]. Using this DE, we can formulate hypotheses for mechanisms of formation of population size oscillations for tundra animals and disclose three main factors determining this, namely, 1) the rate of biomass growth in a favorable year; 2) the maximal population size; 3) survival in the most adverse conditions (or two dimensionless parameters: relative population growth rate and portion of assuredly surviving animals). The first factor characterizes the balance between the fertility and mortality processes in the absence of 'environmental pressure'; the second one characterizes the ecosystem as a whole and reflects the coevolution of lemmings and feed base; the third parameter characterizes adaptation features of lemmings in extreme conditions and is mainly determined by local characteristics, in particular, by the land relief at the place of overwintering [1, 11, 13].

These studies of DE get particular relevance due to the fact that the simulated population of lemmings of Western Taimyr has typical oscillation of the population size maxima with the period of three years [10], whereas a cycle of period three in the order of Sharkovsky guarantees the existence of cycles of any length [12]. In this paper we propose a DE and a scenario of variations of a chosen parameter so that this provides the change of cycle periods in the order of the natural series.

1. Mathematical model

1.1. Justification of the use of the difference equation

As the result of undertaken studies of the tundra 'vegetation-lemmings-Arctic fox' community [1, 11, 13], we get difference equation (1.1) connecting the lemming population sizes for two adjacent years. Using this equation, we reproduced the time dynamics qualitatively close to the dynamics of real population of lemmings (see Fig. 1).

For the normalized variable $\tilde{L} = L/L_{\max}$ it has the form

$$\tilde{L}_{n+1} = \begin{cases} P\tilde{L}, & \tilde{L}_n \leqslant 1/P \\ 1 - r(\tilde{L} - 1/P), & 1/P < \tilde{L}_n \leqslant 1/P - (1-d)/r \\ d, & 1/P - (1-d)/r < \tilde{L}_n \leqslant 1. \end{cases} \tag{1.1}$$

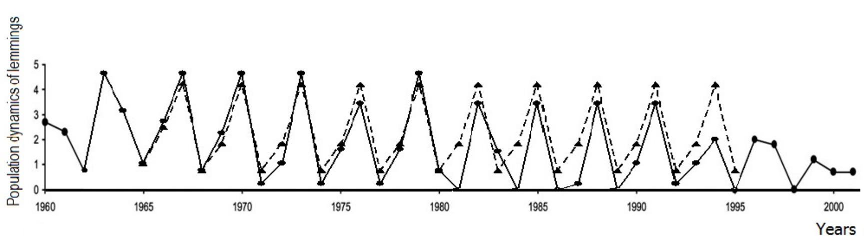

Figure 1. Comparison of the experimental data (Taimyr Peninsula) [10] (solid line) and calculations for the population dynamics with the use of difference equation (1.1) (dotted line) [13].

Here P is the increment of the biomass of lemmings in a favorable year; d is the normalized biomass of lemmings in the optimal biotope, the coefficient r characterizes the change in the biomass of lemmings in the case of insufficient food in spring.

It should be noted that the value d (the capacity of the optimal biotope) is most hard to estimate among all three parameters. Therefore, it is natural to take this parameter as a bifurcation one, determine domains where the behaviour of trajectories weakly depends on this factor, and identify domains where the sensitivity to variations of this factor is high.

The results of computational experiments with equation (1.1) for $P = 2$ and $r = 100$ are presented in the bifurcation diagram in Fig. 2 [9]. The character of dynamic modes was studied for the range of the parameter d varying from 1 to 0. In Fig. 2 we can see stability zones separated by transitional zones with complex modes (black vertical strips).

The following assertion is valid. In the case of equation (1.1) and if the parameter d varies from 1 to 0, stability zones appear successively and transitional zones with complex modes separate them. The trajectories have constant periods inside the stability zones and in the transition from one stability zone to the other the period changes in the order of the natural series. Each transition zone has periodic trajectories with the period exceeding an arbitrary natural number given in advance. In this case the 'width' of transitional zones can be made arbitrarily small when the parameter r tends to infinity.

For a given value d the period of a trajectory is visually estimated in the following way: we draw a vertical line across the fixed point with the value d on the abscissa axis and the number of intersections of this vertical line with the trajectory determines the period of the trajectory for this d.

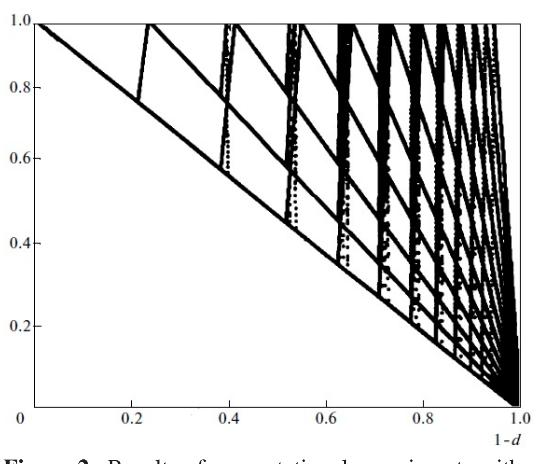

Figure 2. Results of computational experiments with model (1.1); the dependence of trajectories of the model on the value 1-d. The abscissa axis corresponds to the value 1-d. For the chosen value of d the vertical cross section of the graph represents the points of the trajectory.

1.2. The study of the difference equation by the method of approximation of implicitly specified sets

We study the considered mapping by its visualization. However, first we should construct it explicitly by methods of approximation of implicitly specified sets [6, 7]. We present here a brief description of the applied technique for the case of approximation and visualization of nonlinear mappings.

A set T is called a metric ε-net for a set S if any point of S is positioned at the distance not greater than ε from a certain point of T. Metric nets allow us to encode infinite completely bounded sets by a finite set of points. If T is a metric ε-net for S, then an ε-neighbourhood of T, i.e., the set $(T)_\varepsilon$ approximates the set S. This means that $T \subset S \subset \{B(x,\varepsilon) : x \in T\}$, where $B(x, \varepsilon)$ is a ball of radius ε with the center at the point x. Thus, we get more accurate approximation for lesser ε, but the greater number of the metric net points is required.

The deep holes method [4] is used for the construction of approximations in the method of reachable sets for nonlinear systems. The rate of convergence of the deep holes method is determined by the metric (fractal) dimension of the approximated set.

The following figures present approximations of different sets. Since we use the Chebyshev metric, metric balls are cubes (squares in the two-dimensional case). Therefore, two-dimensional sets consist of the small squares. Three-dimensional sets consist of the small cubes. The figures show their two-dimensional cross-sections, each section is drawn with its own color. The visualization uses the interactive decision maps technology [7].

Let us consider difference equation (1.1) connecting the normalized population sizes of lemmings in two adjacent years in more detail.

We denote the normalized population size of lemmings in (1.1) by $Y(t)$. We study the trajectory beginning from the 'plateu' d, i.e., study the properties of a stable cycle of system development, $Y(0) = d$. For $t = 1, 2, \ldots$ we calculate $Y(t+1)$ for $Y(t)$ according to formula (1.1). We are interested in various characteristics of this trajectory, i.e., the set of reachable states of the system $Y(t)$ for different t, cyclic structures of trajectories, their periods, etc. for different combinations of values of the parameters P, r, and d.

We are interested in the following *sets and indicators*. First of all, in the space of the parameter d and the phase Y we consider the set

$$\mathbf{A}(P, r, d) = \{(d, Y(t)), t = 0, \ldots, N\}.$$

Hereafter, we assume $N = 1000$. We construct and study the sets $\mathbf{A}(P, r, d)$ for all $d \in [0, 1]$ and obtain:

$$\mathbf{A}(P, r) = \{\mathbf{A}(P, r, d) : d \in [0, 1]\}.$$

For large N these sets can be assumed as approaching the attractor of the considered

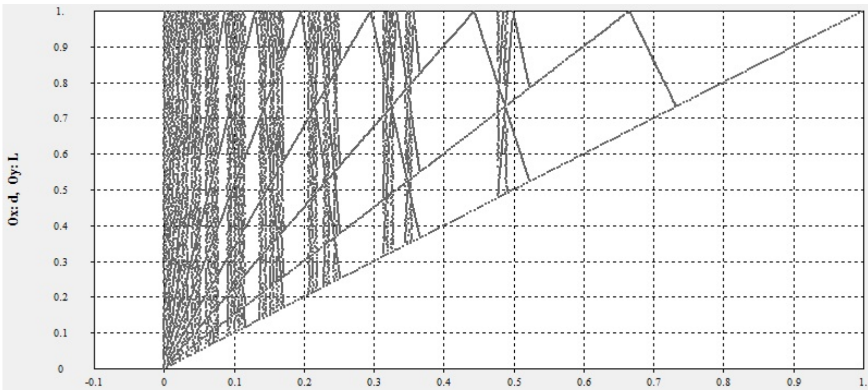

Figure 3. Bifurcation diagram of **A** for P=1.5, r=4.

system.

We are also interested in the value characterizing the periodicity of the considered trajectory, i.e.,

$$\text{Period}\,(P,\ r,\ d) = \min\,\{t\text{: —}Y(t) - d\text{—} \leqslant \Delta,\ t = 1, \ldots, N\}$$

where Δ is the given accuracy of the cycle 'closure'. Unless otherwise specified, below we assume $\Delta = 10^{-8}$.

We are also interested in the sets

$$\mathbf{B}(P,\ r,\ d) = \{(d,\ Y(t),\ \text{Period}(P,\ r,\ d)),\ t = 0, \ldots, N\}$$

$$\mathbf{B}(P,\ r) = \{\{\mathbf{B}(P,\ r,\ d)\text{: } d \in [0,\ 1]\}$$

which represent the graph of the cycle length given at the points (d, Y) of the attractor **A** (P, r). We also consider maps of the cycle period as functions of the parameters P and d (for fixed r), i.e., the sets

$$\mathbf{C}(r) = (P{,}d,\ \text{Period}(P,\ r,\ d))$$

The following bifurcation diagram (Fig. 3) shows the approximation of the set **A**(1.5, 4)={**A**(1.5, 4, d): $d \in [0, 1]$} with the use of metric ε-nets. In the figures the abscissa axis corresponds to the parameter d and the ordinate axis corresponds to the set of values $Y(t)$ for $t = 0, \ldots, N$.

This figure presents the same properties of the increasing chaotic behaviour of the system when the parameter d tends to 0 as this was indicated in the previous section.

Figure 4 presents the approximation of the set $\mathbf{B}(1.5,4) = \{\mathbf{B}(1.5,4,d) : d \in [0.01,1]\}$ by metric ε-nets. The set **B** is a stratification of the set **A** in the indicator

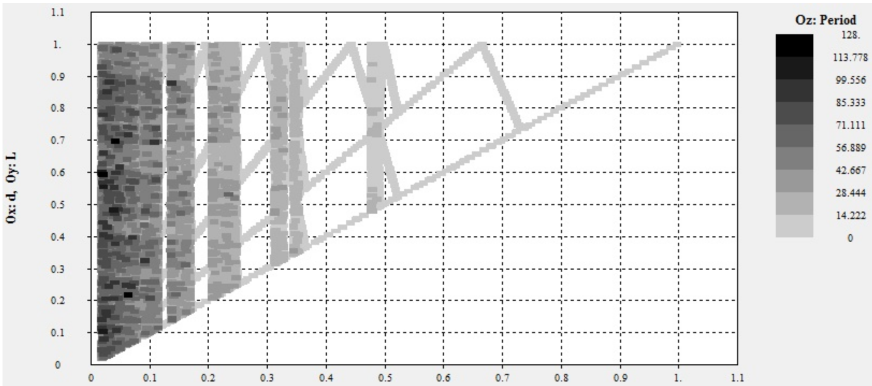

Figure 4. Stratification **B** of the bifurcation diagram **A** over the period of the cycle for P=1.5, r=4.

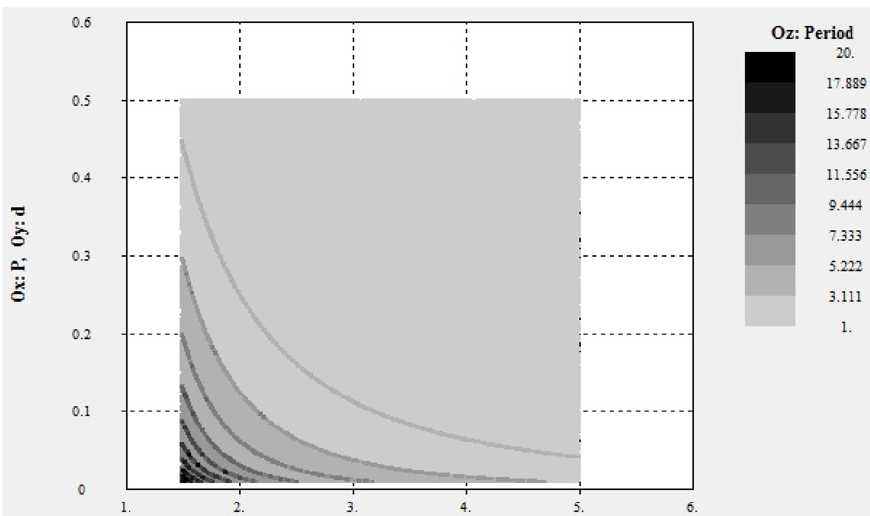

Figure 5. Map of the cycle period **C** for $r = 100$.

Period, i.e., the graph of this function at the points of **A**.

The abscissa axis in the figure corresponds to the parameter d and the ordinate axis corresponds to the set of values $Y(t)$ for $t = 0, \dots, N$. The color (density of hatching) corresponds to the value of the indicator Period(d). The same color may indicate different values of Period. The palette of colors is presented at the right side of Fig. 4.

The figure shows that if the value d tends to zero, then we get trajectories with an arbitrarily large period. In this case, only a part of the graph is presented for $d \geqslant 0.01$, therefore, the maximal value of the cycle period for these values of the parameters P and r and the accuracy Δ of closure of the cycle does not exceed 128.

Figure 5 shows the map of the set **C**(100), i.e., for $r = 100$, as a set of graph layers for different ranges of graph periods. The correspondence of period values to colors (to hatching in monochrome images) is presented at the right side of Fig. 5.

In Fig. 5 the abscissa axis corresponds to the parameter P and the ordinate axis corresponds to the parameter d. The color indicates the range of values of Period(d) corresponding to that color. For example, the plateau in the upper right part characterizes combinations of these two parameters so that the period equals 1, 2, or 3. In monochrome images each color is represented by hatching of specific density. One can see 'mountain ridges' of points with large periods (in these figures they are darker).

Figure 5 presents only the points with $P \in [1.5, 5]$ and $d \in [0.01, 0.5]$. It is seen that, approaching the lower left corner (the point with the minimal values of parameters), the value of the basic period sharply increases and 'ridges of large periods' become more frequent.

Figure 6 presents the map of the set $\mathbf{C}(10)$, i.e., for $r = 10$. The figure shows that if the parameters P and d decrease, then the period tends to infinity, i.e., for small P and d we get a singularity of the period characterized by the concept of 'blue sky singularity' [3], i.e., there are no images of points with the cycle period exceeding 1000 in the lower left corner. In this case the front of the 'blue sky disaster' is most visible.

In conclusion we present the study of the set of existence defined here as the set of triples of combinations for values of the main parameters (P, d, r) such that the period is within a given range. For example, we can consider the set of 'resonance' combinations of parameters such that the cycle period is greater than 5. This set is shown in Fig. 7.

In Fig. 7 the abscissa axis corresponds to values of the parameter P and the ordinate axis corresponds to the parameter d. The color (hatching) indicates the values of the parameter r according to the palette in the right side of the figure. Recall that for all these combinations of parameters the cycle period of a trajectory does not exceed 5.

It is seen from Fig. 7 that the considered set of existence lies in the domain of small values of the parameters P and d. The figure also shows the structure of this set. It is formed by three-dimensional fragments (in the space of the parameters P and d those have a hyperbolic form) slanted in the third dimension to the direction of decrease of these two parameters for increasing r.

2. Analytic study of difference equations

In addition to the study of obtained mapping (1.1) with the use of computational experiments, we can also use analytic (non-computer) methods. To do that, let us consider the non-concave one-dimensional unimodal mapping (1DUM) $X^{t+1} = F(X^t)$ of the segment [0,1] onto itself possessing the following properties.

The function F monotonically increases on the segment [0, D] (D ¡ 1) and attains its maximal value at the point D and then it decreases passing through the fixed point (equilibrium position) $A = F(A)$ not equal to (0,0).

Examples of such equations are above equation (1.1) and also the triangular mapping equation $X^{t+1} = F_0(X^t) = 1 - 2|0.5 - X^t|$.

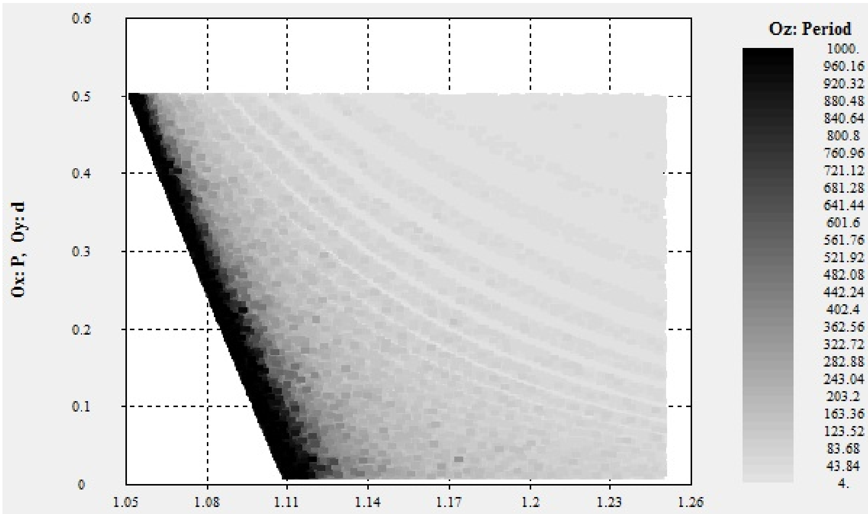

Figure 6. Map of the cycle period for **C** for r=10 in the region of small values of the parameters P and d ('blue sky' singularity).

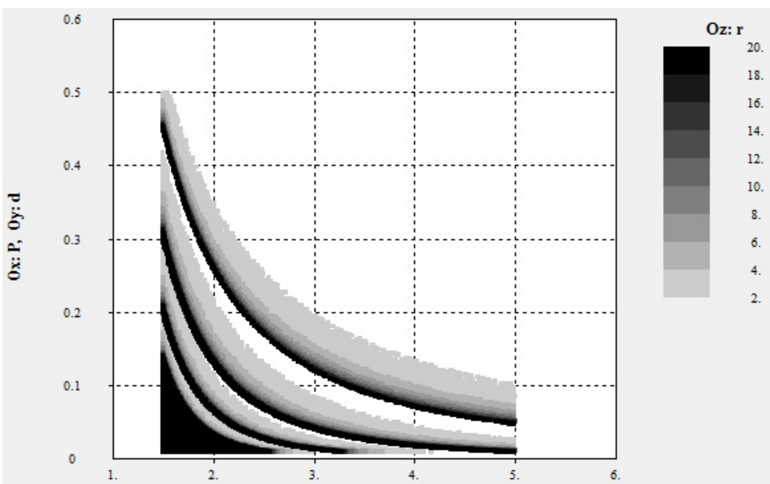

Figure 7. The set of the 'resonance' parameters P, d, and r such that *Period¿*5.

To analyze behaviour of trajectories, we define two following sets of points: $G = \{A_n, n = 0,1,2,\dots\}$, $K = \{D_n, n = 0,1,2,\dots\}$. The set G consists of points A_i such that $F^i(A_i) = A$, the set K consists of points D_i such that $F^i(D_i) = D$, where $F^i(\cdot) = F(F\dots(F()))$ is a i-fold mapping. In this case, $A_0 = A, D_0 = D$. If the trajectory is to the left of the point A and the ordinate of one its point falls into the interval $[A_n, A_{n-1}]$, then at the next cycle the trajectory falls into the interval $[A_{n-1}, A_{n-2}]$.

Basic definitions. In order to analyze the results of computational experiments with lowering the plateu d (see Fig. 2), we proposed constructive techniques to obtain periodic trajectories of 1DUM and introduced the corresponding methods of study [9]. These techniques can be used for a wide class of mappings $X^{t+1} = F(X^t)$.

The equilibrium position A divides the segment [0, 1] into the two domains [0, A] and [A, 1]. These parts are unequal. The right-hand part of a trajectory cannot have two cycles in succession; it is some kind of 'reflector' actually specifying initial values for the movement of the trajectory in the left-hand part of the function, and in the left part of the trajectory it can have several cycles.

The set reflects the specificity of 1DUM, it shows what number of cycles is in the domain [0, A], determines the distance between maxima inside the trajectory. It possesses the following evident property: for any natural number n there exists a neighbourhood of zero such that for $A_{n-1} < X^0 < A_n$ the trajectory is in the left-hand side of the domain $(X^t < A)$ for exactly n cycles and then it passes to the right-hand side $(X^t > A)$. There exists a wide class of functions for which we can take a scenario leading to the situation that in most cases the set $\{A_n\}$ determines the character of trajectories including the length of the cycle.

In some cases it is sufficient to use the set $\{A_n\}$ for practical problems of study of possible dynamic modes in discrete mapping. In particular, in the case when the degree of reliability of biological information allows us to analyze only time intervals between the maxima of population size [10].

For more detailed study of properties of the considered mappings we introduce the additional construction called line return (LR).

Definition 2.1. A line return of nth order (LRn) for a mapping F is the curve in the rectangle $A \leqslant X^t \leqslant 1;\ 0 \leqslant X^{t+1} \leqslant A$ being the graph of the function $F_c^{(n)}(X^{t+1})$ mapping the segment $0 \leqslant X^{t+1} \leqslant A$ onto the segment $A \leqslant X^t \leqslant 1$ according to the following algorithm.

Construction algorithm for LRn. We draw a horizontal line through any value X^{t+1} from the segment $0 \leqslant Xt + 1 \leqslant A$ in the rectangle $A \leqslant X^t \leqslant 1; 0 \leqslant X^{t+1} \leqslant A$. Then we take any point X^{t+1} from the segment $0 \leqslant X^{t+1} \leqslant A$ and draw a horizontal line through it in the rectangle $A \leqslant X^t \leqslant 1;\ 0 \leqslant X^{t+1} \leqslant A$. The point of intersection with the graph of the original function to the right of the equilibrium position (EP) gives the initial point. We construct the trajectory graphically by using Lamerey's ladder. At the nth return to the right of the equilibrium position over the bisectrix of the angle between the abscissa and ordinate axes we draw the corresponding vertical line. The point of intersection of this line with the testing horizontal line with the coordinates (X^t, X^{t+1}) belongs to LRn. Apply the similar procedure to all points X^{t+1} from the segment [A, 1] and join all points of intersection. As the result, we obtain LRn. Thus, we have associated each value X^{t+1} with the value X^t in the indicated rectangle and hence define the function X^t = LRn (X^{t+1}). The graphical implementations of this construction algorithm for LR1 and LR2 for the triangular mapping are presented in Figures 8 and 9, respectively.

The algorithm implies another definition of LR. Let us suppose a segment of

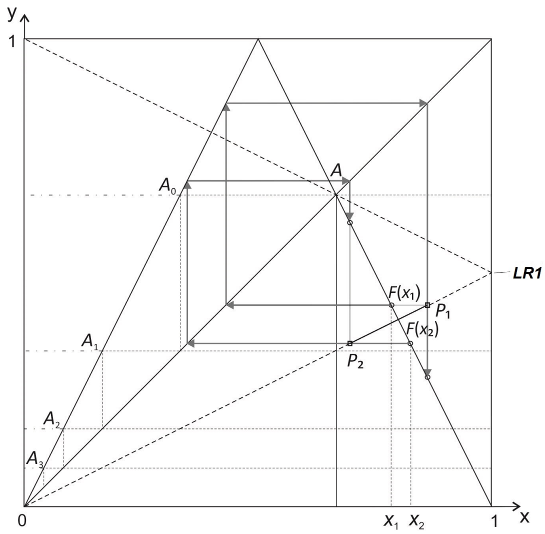

Figure 8. Construction of lines of the first return. Trajectories passing through the points $F(x_1)$ and $F(x_2)$ give the points P_1 and P_2 forming the segment of first line return (LR1).

a trajectory such that its first and last points are positioned after EP. In this case the line return of the kth order can be defined as the locus of points such that their coordinate X^t equals the coordinate X^t of the last point, and their coordinate X^{t+1} equals the coordinate X^{t+1} of the first point of that segment. In this case k is the number of returns of the trajectory to the right of EP in this segment.

Proposition 2.1. *The points of intersection of LRn with the graph of the initial function F (GIF) determine periodic trajectories. In this case, using LRn, we can obtain all periodic trajectories with the period less than or equal to n.*

Definition 2.2. The domain from the point A to the point A_1 is called the zone of two, the domain from the point A_1 to the point A_2 is called the zone of three, etc.

The number of a zone determines the number of cycles required for a trajectory to fall to the right of EP again, and it also determines the period of the cycle passing through the point of intersection of the graph of the initial function and LR1.

Proposition 2.2. *Lines of LR can be constructed as fragments of F^n specularly rotated by* 90°.

Proof. Take an arbitrary point on LR formed by F^{n-1} specularly rotated relative to the bisectrix by 90° and draw a horizontal line to the bisectrix. At this point we have the value F^{n-1}. According to Lamerey's algorithm, we draw a horizontal line to the bisectrix and then draw a vertical line from the point of their intersection. This vertical line passes through the original point constructed by the specular rotation

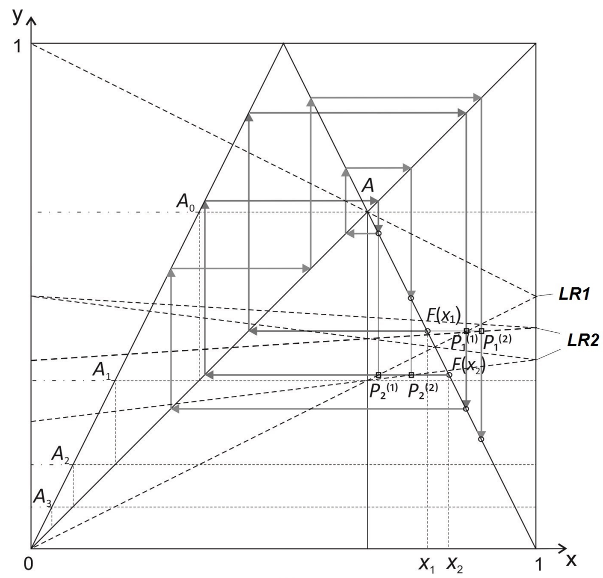

Figure 9. Construction of lines of the second return. Trajectories passing through the points $F(x_1)$ and $F(x_2)$ in the first return in the domain to the right of the equilibrium position give the points $P_1^{(1)}$ and $P_2^{(1)}$ forming a segment of the first return line ($LR1$). The points $P_1^{(2)}$ and $P_2^{(2)}$ are obtained in the second return and belong to different segments of LR2.

of F^{n-1}.

Proposition 2.3. *The points of intersection of LR with GIF constructed as fragments of F^{n-1} specularly rotated by* 90° *lie on periodic trajectories of period n.*

Proof. We repeat arguments similar to above ones. Take any point of intersection of LR and GIF and draw a horizontal line to the bisectrix. This point corresponds to some value of the function F^{n-1}. According to Lamerey's algorithm, we draw a horizontal line from this value to the bisectrix and then draw a vertical line from the point of their intersection to the intersection with GIF. At the point of intersection we have LR formed by F^{n-1} specularly rotated by 90°. Thus, we have got a cycle of period n.

Proposition 2.4. *The points indicates above form the complete set of points of period n lying to the right of the equilibrium position (in the zone of formation of LR).*

Proof. Let us take an arbitrary point of period n positioned to the right of EP. According to Proposition 2.2, it must lie at the point of intersection of some LR with GIF. Draw a horizontal line from this point to the bisectrix. This point corresponds to some value of the function F^{n-1}. According to Lamerey's algorithm, we draw a horizontal line from this value to the bisectrix and then draw a vertical line from the point of their intersection to the intersection with GIF. At the intersection we have the original point. Since the point was chosen arbitrarily (from the corresponding points lying to the right of EP), Proposition 2.4 is proved.

Proposition 2.5. *Let n be some period. In the triangular mapping (TM) the cycle points of period n lying to the right of the equilibrium position (in the zone of LR formation) are defined as the points of intersection of LR with the graph of the initial function. The equations determining their coordinates have the form*

$$X^{t+1} = \begin{cases} \dfrac{4i-2}{2^n+1} & \text{for a rising,} \quad i = 1, \dots, 2^{n-2} \\ \dfrac{4i-2}{2^n-1} & \text{for a downward slope,} \quad i = 1, \dots, 2^{n-2}. \end{cases} \tag{2.1}$$

The index i in this formula indicates the ordinal number of a prong among the sequence of prongs of LR formed by the function F^{n-1} specularly rotated by 90º. The enumeration begins with the most lower prong (closest to the abscissa axis).

2.1. The study of transitional modes

In order to study transitional modes, we use the triangular mapping (TM) supplemented by a plateu d to the right of the equilibrium position (EP). This mapping is a particular case of mappings defined by formula (1.1). It has wide transitional zones, a high level of symmetry, and so we can analyze it easily. We applied the 'bifurcation study' for TM, i.e., determined cycles appearing in the process of lowering the plateu. The LR technique is ideal for analysis of results of computational experiments with lowering the plateu. If the plateu is at a certain place, then it is crossed by LRn, and the LR with the least number among those LRn that lie above the graph of the initial function is realized. Thus, the analysis of the sequence of appearing cycles in the process of lowering the plateu is reduced to the study of changes in minimal numbers of LRn that are above the graph of the initial function. In order to determine such numbers, we use the following procedure: we successively consider LR with the increasing number n.

In the case of triangular mapping with lowering the plateu, formulas (1.1) and (2.1) imply the following result.

Proposition 2.6. *The coordinates bounding the domain of realization of a cycle with a given period for the triangular mapping with lowering the plateu are calculated by the formulas*

$$X^{t+1} = \begin{cases} \dfrac{4i-2}{2^n+1} & \text{for boundaries nearest to EP,} \quad i = 1, \dots, 2^{n-2} \\ \dfrac{4i-2}{2^n-1} & \text{for boundaries far from EP,} \quad i = 1, \dots, 2^{n-2}. \end{cases}$$

Proposition 2.6 and formulas forming this assertion imply Propositions 2.7–2.9.

Proposition 2.7. *If multiple cycles exist, then all cycles with lesser periods surely contact a multiple cycle of greater length.*

Propositions 2.6 and 2.7 imply the following result.

Proposition 2.8. *If a cycle of some period n appears in the procedure of successive increase of cycle periods, then cycles with the periods* $n2^m$, $m = 1,2,3,\ldots$, *appear directly after it.*

Since the domains of realization of doubling cycles are 'glued', we have proved the following.

Proposition 2.9. *Inside the sequence of cycles* $n2^m$, $m = 1,2,3,\ldots$, *there are no cycles of other periods.*

Proposition 2.10. *There are no cycles of periods* $n2^m$, $m = 1,2,3,\ldots$, *directly before any cycle of period n.*

Analogues of Feigenbaum's assertions. Feigenbaum [2] presented assertions on relations between the sizes of cycle realization domains in the process of doubling of cycle period ($n = 2,4,8,16,\ldots$).

Studying bifurcation modes in TM appearing under variations of the parameter d ('plateu height'), we have formulated more general assertions.

Proposition 2.11. *The following relation is valid for the sizes (width) of adjacent domains where cycles of periods n and 2n are realized:* $(2^{2n} - 1)/(2^n - 1)$.

It should be noted that for n tending to infinity the reverse ratio of sizes of adjacent domains tends to zero, i.e., $(2^n - 1)/(2^{2n} - 1) \to 0, n \to \infty$. Obviously, the value of this ratio is equivalent to $1/2^n$.

In contrast with Feigenbaum's relation, this one holds for any generating number $n_* = 2,3,4,5,\ldots$.

Let us study in detail the domain between first appearances of cycles with the periods 4 and 6. We have the following result.

Proposition 2.12. *All even cycles lie between the intervals of the domain of parameters where cycles of periods 4 and 6 appear for the first time.*

Successively considering LR with increasing numbers n in the domain between the first appearance of cycles with the periods 4 and 6, we come to the following sequence of cycles.

Up to period 8: 4, 8, 6;

Up to period 10: 4, 8, 10, 6;

Up to period 12: 4, 8, 12, 10, 6;

Up to period 14: 4, 8, 12, 14, 10, 14, 6;

Up to period 16: 4, 8, 16, 12, 16, 14, 10, 14, 16, 6;

Up to period 18: 4, 8, 16, 12, 16, 18, 14, 18, 10, 18, 14, 18, 16, 6;

Up to period 20: 4, 8, 16, 20, 12, 20, 16, 20, 18, 14, 18, 10, 20, 18, 14, 18, 20, 16, 20, 6;

Up to period 22: 4, 8, 16, 20, 12, 20, 16, 20, 22, 18, 22, 14, 22, 18, 22, 10, 20, 22, 18, 22, 14, 22, 18, 22, 20, 16, 20, 22, 6;

Up to period 24: 4, 8, 16, 24, 20, 12, 24, 20, 24, 16, 24, 20, 24, 22, 18, 22, 14, 22, 18, 22, 24, 10, 20, 24, 22, 18, 22, 24, 14, 24, 22, 18, 22, 24, 20, 24, 16, 24, 20, 24, 22, 6;

Up to period 26: 4, 8, 16, 24, 20, 12, 24, 20, 24, 16, 24, 20, 24, 26, 22, 26, 18, 26, 22, 26, 14, 26, 22, 26, 18, 26, 22, 26, 24, 10, 20, 24, 26, 22, 26, 18, 26, 22, 26, 24, 26, 14, 26, 24, 26, 22, 26, 18, 26, 22, 26, 24, 20, 24, 26, 16, 26, 24, 20, 24, 26, 22, 26, 6;

Up to period 28: 4, 8, 16, 24, 28, 20, 28, 12, 24, 28, 20, 28, 24, 28, 16, 28, 24, 28, 20, 28, 24, 28, 26, 22, 26, 18, 26, 22, 26, 14, 28, 26, 22, 26, 18, 26, 22, 26, 28, 24, 28, 10, 20, 28, 24, 28, 26, 22, 26, 28, 18, 28, 26, 22, 26, 28, 24, 28, 26, 14, 28, 26, 28, 24, 28, 26, 22, 26, 28, 18, 28, 26, 22, 26, 28, 24, 28, 20, 28, 24, 28, 26, 28,16, 28, 26, 28, 24, 28, 20, 28, 24, 28, 26, 22, 26, 28, 6.

These sequences of alternation of cycles under the change of the bifurcation parameter in the considered interval do not contradict the order of Sharkovsky. Using Propositions 2.8–2.10, we can formulate the following result.

Proposition 2.13 similarity hypothesis. *If a certain sequence of alternation of even cycles is realized (in the procedure of successive increase of periods)* $2a_i$*, then this sequence is realized under the replacement of 2 by* 2^m*,* $m = 1,2,3,\dots$*, i.e., the sequence* $2^m a_i$ *is realized.*

The similarity hypothesis was checked by comparison of the corresponding sequences of cycles.

3. Conclusions

Let us sum up the results of performed researches. The study of models of tundra community revealed the leading role of the population of lemmings in formation of animal population fluctuations in this community [1, 11, 13]. It was possible to use difference equations (DE) to describe this population. Such equations have a non-traditional from (different from the widely known logistic one) [2, 5, 8]. We should note that not only the form of the equation, but the choice of the bifurcation parameter d (the capacity of optimal biotope) were justified as well. In this regard, it is natural to take this parameter as a bifurcation one. In the analysis of mappings we used the method of approximation of implicitly specified sets and interactive decision maps [4, 6, 7].

Computational experiments allowed us to obtain domains where the behaviour of trajectories weakly depends on the chosen indicator and also reveal the domains where a high sensitivity to variations of this indicator is typical. In this case, passing from one stability zone to another, the period of oscillations changes in the order of the natural series (1, 2, 3, 4,. . .), whereas stability zones are separated by transitional zones with more complicated dynamic modes [9]. An important aspect of conducted investigations is the study of these transitional zones ('ordering of the chaos') where cycles of large periods alternate with a high density and their values change for small variation of the bifurcation parameter. Traditional (computer) studies of the logistic equation (that usually get universal usability [2]) do not concentrate attention on the presence of stability and transitional zones. (Although those can be indicated in results of numerical experiments, see, e.g., [2]). Texts of various origins (see the references in [2]) are roughly the following: first a cascade of doubling of cycle periods occurs, after that the dynamics complicates, and, obtaining a cycle of period three, there exist cycles of all periods (which is interpreted in [5, 8] as the appearance of chaos).

The use of DE formed by segments of straight lines allows us to transform the problem of determination of the sequence of cycle appearances to a 'computable' form, which essentially supports the results of computational experiments. The use of the line return technique helps essentially in the study of transitional zones [9]. This technique gives an algorithm of study for the sequence of appearances of cycles of arbitrarily large period due to a justified method of successive increase of periods of studied cycles. The fact that we can obtain the sequence of appearances of cycles as the result of calculations is a constructive example of extension of Sharkovsky's order [12], which is hard to obtain for the logistic mapping.

After appearance of the period three (as is shown in the example considered here) there are no 'stalling into chaos', but its own patterns and stability domains of cycles appear (with periods greater than or equal to four). All these refinements were checked by computational experiments and partly by analytic calculations.

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