

# Mathematical modelling of oscillatory fluid flow in a tube with deformable walls

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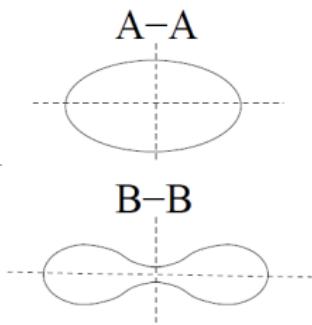
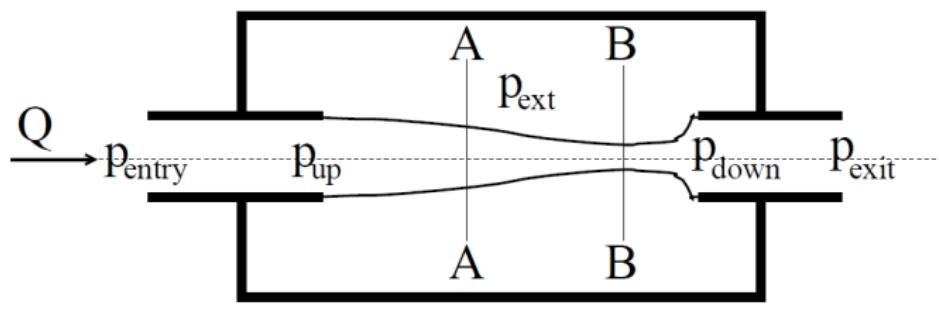
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# Starling resistor

Fluid flow through a Starling resistor:<sup>1</sup>



<sup>1</sup>M. Heil, O. E. Jensen (2003). *Flows in deformable tubes and channels: Theoretical models and biological applications*.

# One-dimensional model

Averaging over cross-section:

$$\frac{\partial A}{\partial t} + \frac{\partial(Au)}{\partial x} = 0,$$

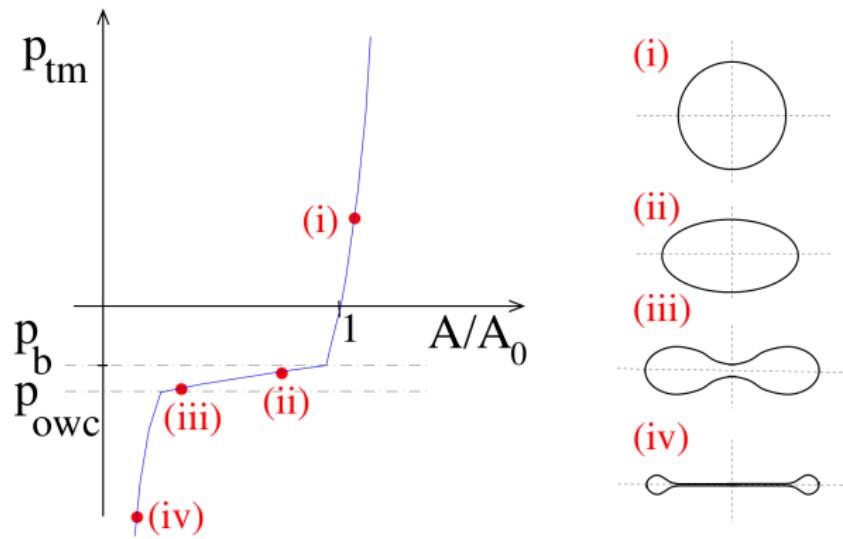
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - Ru,$$

where  $A$  is the cross-sectional area,  $u$  is the mean velocity,  $p$  is the transmural pressure ( $p - p_{\text{ext}}$ ).

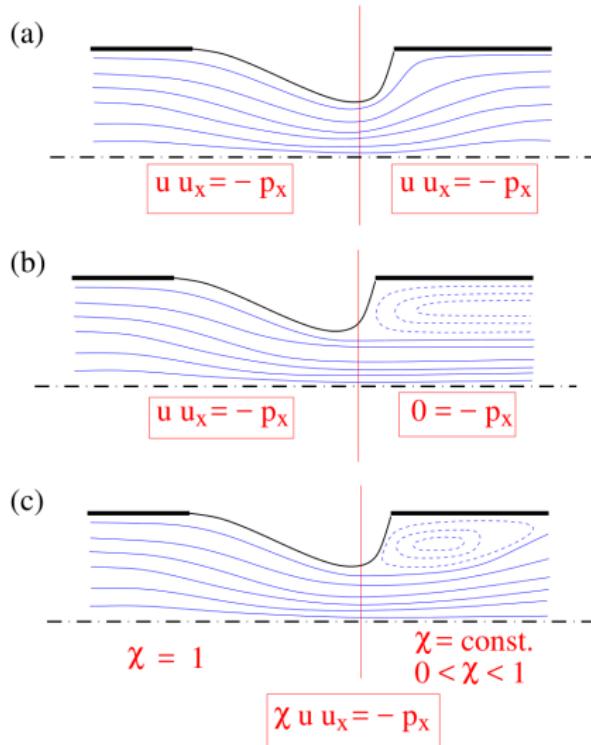
The “equation-of-state” (“tube law”):

$$p = p(t, x, A(t, x)).$$

# Tube law



# Self-excited oscillations



(a) Attached flow; no pressure loss.  
Choking.

(b) Ideal separation; no pressure recovery. Steady flow.

(c) An intermediate case: partial pressure recovery.  
Self-excited oscillations.  
[Cancelli, Pedley, 1985]

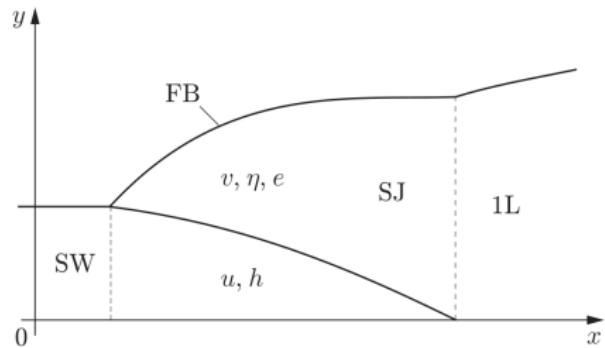
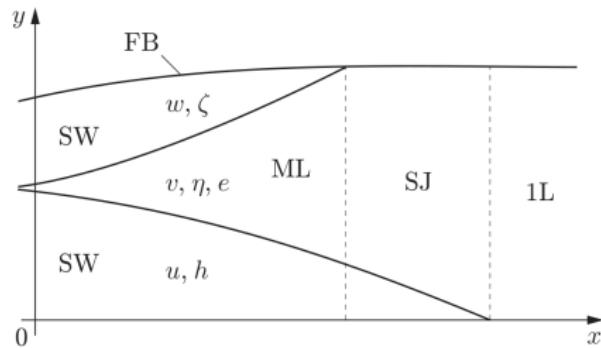
# Outline

- 1 Mathematical model
- 2 Numerical experiments

## 1 Mathematical model

## 2 Numerical experiments

# Multi-layer models



Water waves: mixing layer, near-surface turbulent layer, turbulent bore.

Gas dynamics: pseudo-shocks.

[Liapidevskii, Chesnokov, et al.]

# Symmetric flow

Symmetric flow:

$$\begin{aligned} u_t + uu_x + vu_y + \rho^{-1} p_x &= 0, \\ v_t + uv_x + vv_y + \rho^{-1} p_y &= 0, \\ u_x + v_y &= 0. \end{aligned} \tag{1}$$



# Long-wave approximation

Long-wave approximation ( $\varepsilon^2 \equiv H_0^2/L_0^2 \ll 1$ ):

$$\begin{aligned} u_t + uu_x + vu_y + \rho^{-1} p_x &= 0, \\ p_y &= 0, \\ u_x + v_y &= 0. \end{aligned} \tag{2}$$

Pressure:

$$p = p(t, x).$$

Vorticity:

$$\omega = -u_y.$$

# Two-layer flow

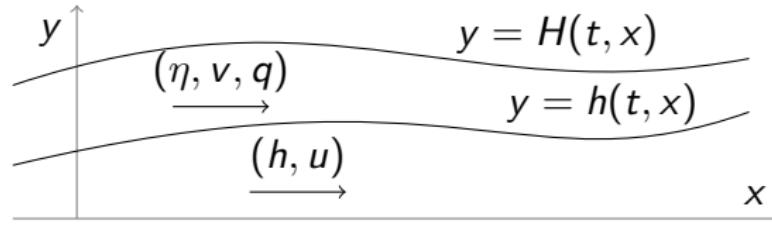
$y \in (0, h)$ : potential core ( $\omega = -u_y = 0$ ).

$y \in (h, H)$ : shear boundary layer.

$H = h + \eta$ : total height

Average velocity in the boundary layer and root-mean-square deviation:

$$v(t, x) = \frac{1}{\eta} \int_h^H u(t, x, y) dy, \quad q^2(t, x) = \frac{1}{\eta} \int_h^H (u - v)^2 dy.$$



# Averaged equations

Conservation of mass:

$$h_t + (hu)_x = 0, \quad \eta_t + (\eta v)_x = 0, \quad (3)$$

Local momentum:

$$u_t + uu_x + \rho^{-1} p_x = 0, \quad (4)$$

Total momentum:

$$(hu + \eta v)_t + (hu^2 + \eta(v^2 + q^2))_x + \rho^{-1} H p_x = 0, \quad (5)$$

Total energy:

$$\begin{aligned} \frac{1}{2}(hu^2 + \eta(v^2 + q^2))_t + \frac{1}{2}(hu^3 + \eta v(v^2 + 3q^2))_x + Q_x + \\ + \rho^{-1}(hu + \eta v)p_x = 0, \end{aligned} \quad (6)$$

where  $Q = \int_h^H (u - v)^3 dy = \mathcal{O}(\varepsilon^{3\beta})$ , if  $u_y|_{t=0} = \mathcal{O}(\varepsilon^\beta)$ ,  $0 < \beta < 1$ .<sup>2</sup>

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<sup>2</sup>V. M. Teshukov (2007). J Appl Mech Tech Phys.

# Governing equations

The governing system of equations is

$$h_t + (hu)_x = -\sigma q, \quad (7)$$

$$\eta_t + (\eta v)_x = \sigma q, \quad (8)$$

$$u_t + uu_x + \rho^{-1} p_x = 0, \quad (9)$$

$$(hu + \eta v)_t + (hu^2 + \eta(v^2 + q^2))_x + \rho^{-1} H p_x = 0, \quad (10)$$

$$\left( \frac{hu^2 + \eta(v^2 + q^2)}{2} \right)_t + \left( \frac{hu^3 + \eta v(v^2 + 3q^2)}{2} \right)_x + \rho^{-1}(hu + \eta v)p_x = -\sigma \kappa q^3 \quad (11)$$

for unknown functions  $h(t, x)$ ,  $\eta(t, x)$ ,  $u(t, x)$ ,  $v(t, x)$ ,  $q(t, x)$ .

Parameters  $\sigma$ ,  $\kappa$  characterize mixing and energy dissipation.

# Characteristics

Non-conservative form:

$$h_t + uh_x + hu_x = -\sigma q,$$

$$\eta_t + v\eta_x + \eta v_x = \sigma q,$$

$$u_t + uu_x + \rho^{-1} p_x = 0,$$

$$v_t + vv_x + 2qq_x + \eta^{-1}q^2\eta_x + \rho^{-1}p_x = \sigma q(u - v)/\eta,$$

$$q_t + qv_x + vq_x = ((u - v)^2 - (1 + \kappa)q^2)\sigma/(2\eta).$$

It can be shown that there exist at least three real characteristic velocities.  
One entropy characteristic  $\lambda = v$  and four characteristics:

$$1 - \frac{bh}{(\mu - \delta)^2} = \frac{b\eta}{(\mu + \delta)^2 - 3q^2},$$

where  $\mu = \lambda - \bar{u}$ ,  $\bar{u} = (u + v)/2$ ,  $\delta = (u - v)/2$ ,  $b = \rho^{-1}dp/dH$ , which has at least two real roots.

# Stationary solutions

The governing system is Galilean invariant. Hence, to study travelling waves we can consider steady-state solutions:

$$\begin{aligned}uh_x + hu_x &= -\sigma q, \quad v\eta_x + \eta v_x = \sigma q, \quad uu_x + \rho^{-1}p_x = 0, \\vv_x + 2qq_x + \eta^{-1}q^2\eta_x + \rho^{-1}p_x &= \sigma q(u - v)/\eta, \\qv_x + vq_x &= ((u - v)^2 - (1 + \kappa)q^2)\sigma/(2\eta).\end{aligned}$$

Let  $\eta = \eta_0 = 0$  and  $u = u_0$  at  $x = 0$ . Then

$$v_0 = u_0/(2 + \kappa), \quad q_0 = u_0\sqrt{1 + \kappa}/(2 + \kappa), \quad h_0 = H_0.$$

Moreover,

$$H_x|_{x=0} = h_x + \eta_x = \frac{\sigma(1 + \kappa)^{3/2}}{2 + \kappa} \frac{\rho u_0^2}{\rho u_0^2 - p'(H_0)}.$$

## 1 Mathematical model

## 2 Numerical experiments

# Numerical scheme

The system of balance laws

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{R}(\mathbf{u})$$

is solved with the Nessyahu–Tadmor central scheme:<sup>3</sup>

$$\begin{aligned}\mathbf{u}_j^{n+1/2} &= \mathbf{u}_j^n - \frac{\lambda}{2} \mathbf{f}'_j + \frac{\Delta t}{2} \mathbf{R}(\mathbf{u}_j^n), \quad \lambda = \Delta t / \Delta x, \\ \mathbf{u}_{j+1/2}^{n+1} &= \frac{\mathbf{u}_j^n + \mathbf{u}_{j+1}^n}{2} + \frac{\mathbf{u}'_j - \mathbf{u}'_{j+1}}{8} - \\ &\quad \lambda (\mathbf{f}(\mathbf{u}_{j+1}^{n+1/2}) - \mathbf{f}(\mathbf{u}_j^{n+1/2})) + \frac{\Delta t}{2} (\mathbf{R}(\mathbf{u}_j^n) + \mathbf{R}(\mathbf{u}_{j+1}^n)).\end{aligned}\tag{12}$$

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<sup>3</sup>H. Nessyahu, E. Tadmor (1990). J Comput Phys 87.

# Tube law

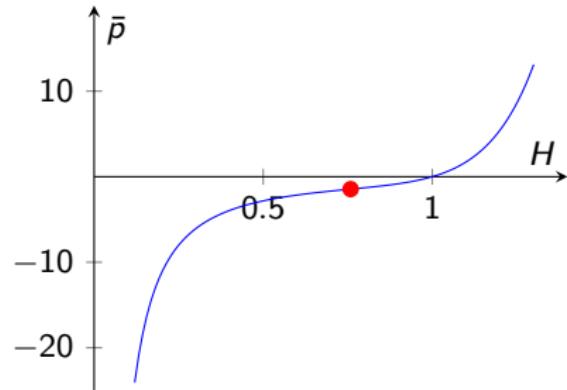
Suppose that the tube law has the following form:

$$p = K(x) \bar{p}(H/H_0).$$

$K(x)$  characterizes the stiffness of the wall,

$$H = h + \eta,$$

$H_0$  corresponds to zero transmural pressure:  $\bar{p}(1) = 0$ .



In the following:

$$\bar{p}(H) = H^{10} - H^{-3/2},$$

$$H_0 = 1.$$

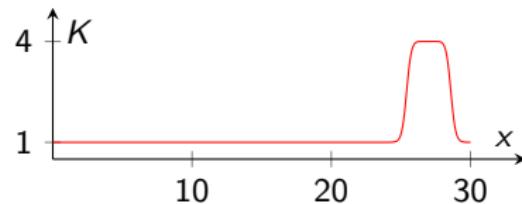
# Numerical experiments

$$L = 30, \quad \sigma = 0.15, \quad \kappa = 6, \quad N = 600.$$

Case 1:

$$H = 1.2, \quad u_0 = 25,$$

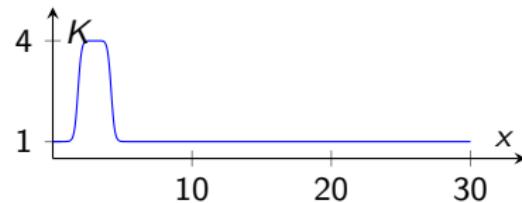
$$K = 1 + 3/(1 + \alpha(x - 0.9L)^8)^2.$$



Case 2:

$$H = 0.8, \quad u_0 = 4,$$

$$K = 1 + 3/(1 + \alpha(x - 0.1L)^8)^2.$$



The system was solved with the 2nd-order Nessyahu–Tadmor scheme.

# Hydroelastic jump

$$H = 1.2, \quad u_0 = 25$$

Smooth hydroelastic jump.

# Oscillatory regime

$$H = 0.8, \quad u_0 = 4$$

Self-excited oscillatory waves of large amplitude.

# Summary

- The model is based on a two-layer flow formulation which is used for description of the pseudo-shocks or smooth transition from supercritical to subcritical flows.
- The mathematical model proposed can describe stationary and non-stationary (self-excited oscillations) regimes.