On the Dynamic Consistency of Some Discrete Predator-Prey Models

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- Another reason of constructing discrete models, at least in case of population model, is that it permits arbitrary time-step units.

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- Standard finite difference schemes sometimes fail to preserve positivity of solutions and dynamic properties of the continuous system.
- Traditional discretization techniques also show numerical instability and exhibit spurious behaviors like chaos.

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- The trivial equilibrium point $x = 0$ is always unstable.
- The non-trivial equilibrium point $x = K$ is always stable for all feasible initial points.

Euler discretization of logistic equation

Euler discrete model of the continuous system (1) is

$$x_{n+1} \quad = \quad x_n + hrx_n(1 - \frac{x_n}{K}), \tag{2}$$

$h(> 0)$ is the step size.

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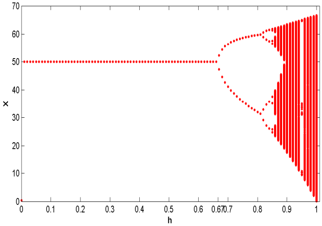

Figure 1.1. Bifurcation diagram of (2) with h as the bifurcation parameter. Here $x(0) = 0.4$, $r = 3$ and $K = 50$.

Decay Equation

Consider the decay equation

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- The corresponding discrete model constructed by Euler forward method is given by

$$x_{n+1} = (1 - \lambda h) x_n. \tag{4}$$

Note that its solution will not be positive if λh is sufficiently large and therefore supposed to show numerical instability.

Dynamics preserving discrete model

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- For such dynamic preserving discretization of the continuous-time system, one can go for **non-standard finite difference (NSFD) scheme** introduced by Mickens during the period 1982-1992 [1].

1. Micken, R. E. (1994). Nonstandard Finite Difference Models of differential Equations; River Edge, NJ: World Scientific.

Some definitions

Consider the differential equation

$$\frac{dx}{dt} = f(x, t, \lambda), \tag{5}$$

where λ represents the parameter defining the system (5). Assume that a finite difference scheme corresponding to the continuous system (5) is described by

$$x_{k+1} = F(x_k, t_k, h, \lambda). \tag{6}$$

We assume that $F(., ., .)$ is such that the proper uniqueness–existence properties holds; the step size is $h = \Delta t$ with $t_k = hk$, $k =$ integer; and x_k is an approximation to $x(t_k)$.

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Definition

Let the differential equation (5) and/or its solutions have a property P. The discrete model (6) is said to be dynamically consistent with the equation (5) if it and/or its solutions also have the property P (Dimitrov and Kojouharov, 2006) [2].

Definition

The finite difference method (6) is called positive if for any value of the step size h, the solution of the discrete system remains positive for all positive initial values (Dimitrov and Kojouharov, 2006) [2].

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The finite difference method (6) is called elementary stable if for any value of the step size h, the fixed points of the difference equation are those of the differential system and the linear stability properties of each fixed point being the same for both the differential system and the discrete system (Dimitrov and Kojouharov, 2006) [2].

2. Dimitrov, D.T. and Kojouharov, H.V. (2006). Positive and elementary stable nonstandard numerical methods with applications to predator-prey models; J. Comput. Appl. Math. 189: 98-108.

Non-standard finite difference(NSFD) scheme

The NSFD procedures are based on two fundamental rules (Mickens, 2005) [3]:

(i) the discrete first-derivative has the representation

$\frac{dx}{dt} \to \frac{x_{k+1}-x_k}{\phi(h)}$,

where $h > 0$ and $\phi(h)$ satisfies the condition $\phi(h) = h + O(h^2)$;

(ii) both linear and nonlinear terms may require a nonlocal representation on the discrete computational lattice. For example,

$$x = 2x - x \to 2x_k - x_{k+1}, \quad x^2 \to x_k x_{k+1}, \quad x^3 \to 2x_k^3 - x_k^2 x_{k+1}.$$

Functional form commonly used for $\phi(h)$ is

$$\phi(h) = \frac{1 - e^{-\lambda h}}{\lambda},$$

where λ is some parameter appearing in the differential equation.

3. Mickens, R.E. (2005). Dynamic consistency: A fundamental principal for constructing NSFD schemes for differential equations; J. Differ. Equ. Appl. 11: 645-653.

Some definitions

Definition

Consider the map $f : R^2 \to R^2$ and let $X^* = (x_1^*, x_2^*)^T$ be a fixed point of f, i.e., $f(X^*) = X^*$. Let λ_1 and λ_2 be the eigenvalues of the variational matrix

$$J(x,y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

evaluated at the fixed point X^*.

The fixed point X^* of the map f is called stable if $|\lambda_1| < 1$, $|\lambda_2| < 1$ and a source if $|\lambda_1| > 1$, $|\lambda_2| > 1$. It is called a saddle if $|\lambda_1| < 1$, $|\lambda_2| > 1$ or $|\lambda_1| > 1$, $|\lambda_2| < 1$ and a non-hyperbolic fixed point if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$ (Elaydi, 2007) [4].

4. Elaydi, S.N. (2007). Discrete chaos with applications in science and engineering; Chapman & Hall/CRC. New York.

Some definitions

Lemma: (Elaydi, 2007) [4] Let λ_1 and λ_2 be the eigenvalues of the variational matrix

$$J(x,y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff the following conditions hold:

$$(i)\ 1 - det(J) > 0,$$

$$(ii)\ 1 - trace(J) + det(J) > 0 \text{ and}$$

$$(iii)\ 1 + trace(J) + det(J) > 0.$$

4. Elaydi, S.N. (2007). Discrete chaos with applications in science and engineering; Chapman & Hall/CRC. New York.

2. Dynamic consistency in a Rosenzweig-MacArthur type predator-prey model with habitat complexity

The continuous time Rosenzweig-MacArthur predator–prey model that considers the effect of habitat complexity is given by Jana and Bairagi [5]:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{\alpha(1-c)xy}{1+\alpha h(1-c)x}, \\ \frac{dy}{dt} &= \frac{\theta\alpha(1-c)xy}{1+\alpha h(1-c)x}-dy. \end{aligned} \tag{7}$$

- α is the maximum attack rate.
- h is the handling time and θ is the conversion efficiency.
- c $(0 < c < 1)$ measures the degree or strength of habitat complexity.
- r and k are the growth rate of prey and environmental carrying capacity, respectively.

5. Jana, D. and Bairagi, N. (2014). Habitat complexity, dispersal and meta populations: macroscopic study of a predator-prey system; Ecological Complexity. 17: 131-139.

Properties of the continuous system (7)

Equilibrium points and existence	Nature of equilibrium points	Stability conditions
$E_0 = (0, 0)$ Always exists	Always unstable	Unconditionally
$E_1 = (K, 0)$ Always exists	Asymptotically stable	$c_0 < c < 1$
$E^* = (x^*, y^*)$ Exists if $c < c_0$, $hd < \theta < 1$	Asymptotically stable Hopf bifurcation Oscillatory coexistence	$c_1 < c < c_0$ $c = c_1$ $0 < c < c_1$
Here $x^* = \frac{d}{\alpha(1-c)(\theta-hd)}$, $y^* = \frac{r(k-x^*)[1+h\alpha(1-c)x^*]}{k\alpha(1-c)}$,	$\frac{hd(\alpha kh+1)}{(\alpha kh-1)} < \theta < 1$, $\alpha > \frac{1+hd}{kh(1-hd)}$,	$c_0 = 1 - \frac{d}{k\alpha(\theta-hd)}$, $c_1 = 1 - \frac{\theta+hd}{\alpha kh(\theta-hd)}$.

NSFD discrete model

For convenience, we first express the continuous system (7) as follows:

$$\begin{aligned} \frac{dx}{dt} &= rx - \frac{r}{k}x^2 - P(x,y)x, \\ \frac{dy}{dt} &= Q(x)y - dy, \end{aligned} \tag{8}$$

where $P(x,y) = \frac{\alpha(1-c)y}{1+\alpha h(1-c)x}$ and $Q(x) = \frac{\theta\alpha(1-c)x}{1+\alpha h(1-c)x}$.
We employ the following nonlocal approximations:

$$\begin{cases} \frac{dx}{dt} \to \frac{x_{n+1}-x_n}{t}, & \frac{dy}{dt} \to \frac{y_{n+1}-y_n}{t}, \\ x \to 2x_n - x_{n+1}, & Q(x)y \to Q(x_{n+1})(2y_n - y_{n+1}), \\ x^2 \to x_n x_{n+1}, & y \to y_{n+1}, \\ P(x,y)x \to P(x_n,y_n)x_{n+1}. & \end{cases} \tag{9}$$

where t (> 0) is the step-size.

NSFD discrete model

By these transformations, the continuous system reads as

$$\begin{aligned} \frac{x_{n+1}-x_n}{t} &= r(2x_n - x_{n+1}) - \frac{r}{k}x_n x_{n+1} - \frac{\alpha(1-c)y_n x_{n+1}}{1+\alpha h(1-c)x_n}, \\ \frac{y_{n+1}-y_n}{t} &= \frac{\theta\alpha(1-c)x_{n+1}(2y_n - y_{n+1})}{1+\alpha h(1-c)x_{n+1}} - dy_{n+1}. \end{aligned} \tag{10}$$

The above system can be rewritten as

$$\begin{aligned} x_{n+1} &= \frac{(1+2rt)[1+h\alpha(1-c)x_n]x_n}{[(1+rt+\frac{rtx_n}{k})\{1+h\alpha(1-c)x_n\}+t\alpha(1-c)y_n]}, \\ y_{n+1} &= \frac{[1+(h+2t\theta)\alpha(1-c)x_{n+1}]y_n}{[(1+td)\{1+h\alpha(1-c)x_{n+1}\}+t\theta\alpha(1-c)x_{n+1}]}, \end{aligned} \tag{11}$$

- Observe that all solutions of the discrete–time system (11) remains positive for any step–size if they start with positive initial values.

Some results of NSFD discrete system (11)

NSFD discrete system (11) has same equilibrium points as in the continuous system (7).

Theorem

(a) The fixed point E_0 is always a saddle point.
(b) The fixed point E_1 is stable if $c_0 < c < 1$, where $c_0 = 1 - \frac{d}{k\alpha(\theta - hd)}$.
(c) The coexistence fixed point E^* is stable if $c_1 < c < c_0$ and $\alpha > \frac{(1+hd)}{kh(1-hd)}$, $\frac{hd(\alpha kh+1)}{(\alpha kh-1)} < \theta < 1$, where $c_1 = 1 - \frac{\theta + hd}{\alpha kh(\theta - hd)}$.
(d) NSFD discrete system (11) undergoes a Hopf bifurcation at $E^*(x^*, y^*)$ when the parameter c^* varies in a small neighborhood of $c = c_1$.

Euler discrete model

Euler discrete system corresponding the continuous system (7) is

$$
\begin{aligned}
x_{n+1} &= x_n + t x_n \Big[r\Big(1 - \frac{x_n}{k}\Big) - \frac{\alpha(1-c)y_n}{1+\alpha h(1-c)x_n} \Big], \\
y_{n+1} &= y_n + t y_n \Big[\frac{\theta\alpha(1-c)x_n}{1+\alpha h(1-c)x_n} - d \Big],
\end{aligned}
\tag{12}
$$

where t is step-size.

- All solutions of the discrete–time system (12) may not be positive for all step–size even if they start with positive initial values.

Some results of Euler system (12)

Euler discrete system (12) also has same fixed points as in the continuous system (7).

Theorem

(a) The equilibrium point E_0 is always unstable.
(b) The equilibrium point E_1 is stable if $c_0 < c < 1$ and $t < min\{\frac{2}{r}, \frac{2\{1+k\alpha h(1-c)\}}{d-k\alpha(1-c)(\theta-hd)}\}$.
(c) Suppose that the interior fixed point E^* exists. It is then locally asymptotically stable if $c_1 < c < c_0$ and $t < min[\frac{G}{H}, \frac{2}{G}]$, where $G = \frac{rx^*}{k\theta}[\theta + hd - kh\alpha(1-c)(\theta - hd)]$, $H = \frac{rx^*}{k}[k\alpha(1-c)(\theta - hd) - d]$.
(d) The Euler system (12) undergoes a Hopf bifurcation at the interior fixed point E^* when $t = \frac{G}{H}$.

Simulation results (continuous system)

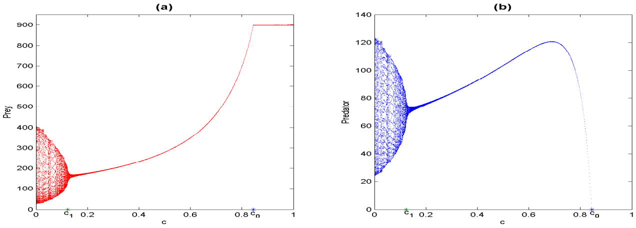


Figure 2.1. Bifurcation diagrams of the continuous system (7) with c as the bifurcation parameter. These figures show that both the prey and predator populations are unstable for $c \in [0, c_1)$ and stable for $c \in (c_1, c_0)$. Prey populations reaches to its carrying capacity and predator populations go to extinction if $c > c_0$. Parameters are $r = 2.65$, $K = 898$, $h = 0.0437$, $\alpha = 0.045$, $d = 1.06$ and $\theta = 0.215$. Here $c_1 = 0.1227$ and $c_0 = 0.8445$.

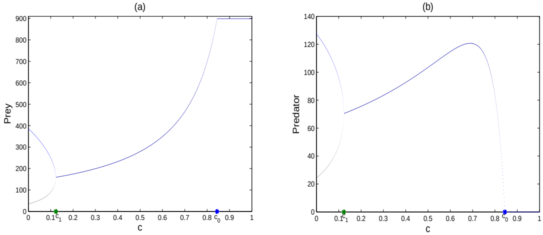


Figure 2.2. Bifurcation diagrams of prey population (Fig. a) and predator population (Fig. b) of the NSFD model (11) with c as the bifurcation parameter with step size $t = 0.1$. These figures show that both the prey and predator populations oscillate for $c \in (0, c_1)$ and stable for $c \in (c_1, c_0)$. The predator–free equilibrium E_1 is stable for $c > c_0$. All parameters are as in the Fig. 2.1.

Simulation results (Euler system)

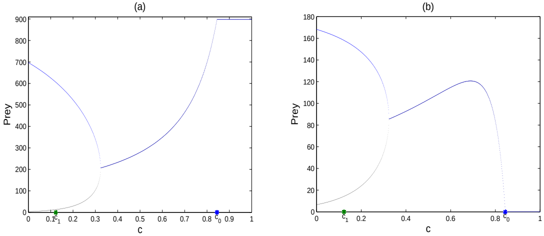


Figure 2.3. Bifurcation diagram of the prey population (Fig. a) and that of the predator population (Fig. b) in Euler's discrete system (12) with step size $t = 0.1$. Parameters are as in the Fig. 2.1. Although parameters of Fig. 2.2 and Fig. 2.3 are same, populations in Fig. 2.3 stabilize at much higher value than c_1, implying dynamic inconsistency of the system (12) with the original continuous system (7).

Simulation results

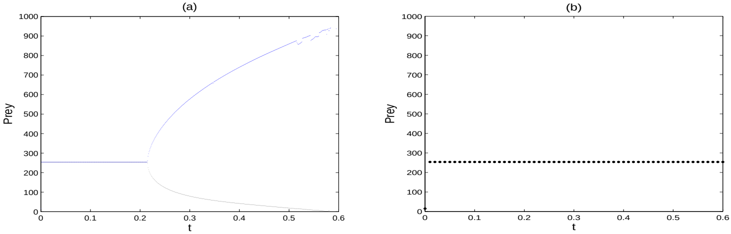


Figure 2.4. Bifurcation diagrams of prey population of the Euler model (Fig. a) and NSFD model (Fig. b) with step–size t as the bifurcation parameter. Here $c = 0.45$ and all the parameters and initial point are same as in Fig. 2.1. The first figure shows that the prey population is stable for small step-size t and unstable for higher value of t. The second figure shows that the prey population is stable all step–size t.

3. A two species competitive system with toxicity

The following coupled nonlinear differential equations are used for two species Lotka–Volterra competition model under the assumption that each species produces a substance toxic to the other only when the other is present (Maynard [7], Chattopadhyay [8], Samanta [9]):

$$\begin{aligned} \frac{dx}{dt} &= x(K_1 - \alpha_1 x - \beta_{12} y) - \gamma_1 xy, \\ \frac{dy}{dt} &= y(K_2 - \alpha_2 y - \beta_{21} x) - \gamma_2 xy, \end{aligned} \tag{13}$$

where x and y are the population densities of two competing species; K_1, K_2 are the growth rates; α_1, α_2 are the intra-specific competition coefficients; β_{12}, β_{21} are the inter-specific competition coefficients; γ_1 and γ_2 are the rates of toxic inhibition of the first species by the second and vice-versa.

7. Maynard, J. (1974). Models in Ecology; Cambridge University Press, Cambridge.
8. Chattopadhyay, J. (1994). Effect of toxic substances on a two-species competitive system; Eco. Modelling. 84: 287-289.
9. Samanta, G. P. (2010). A two species competitive system under the influence of toxic substances; Appl. Math and Computation. 216: 291-299.

Properties of the continuous system (7) (Maynard [7], Chattopadhyay [8], Samanta [9])

Equilibrium points	Existence condition	Stability conditions
$E_0 = (0, 0)$	Always exists	Always unstable
$E_1 = (\frac{K_1}{\alpha_1}, 0)$	Always exists	$\frac{\alpha_1}{\beta_{21}} < \frac{K_1}{K_2}$
$E_2 = (0, \frac{K_2}{\alpha_2})$	Always exists	$\frac{K_1}{K_2} < \frac{\beta_{12}}{\alpha_2}$
$E^* = (x^*, y^*)$ $x^* = \frac{B_1 + \sqrt{B_1^2 + 4(\gamma_2\alpha_1 - \gamma_1\beta_{21})(K_1\alpha_2 - K_2\beta_{12})}}{2(\gamma_2\alpha_1 - \gamma_1\beta_{21})}$, $y^* = \frac{B_2 + \sqrt{B_2^2 + 4(\gamma_1\alpha_2 - \gamma_2\beta_{12})(K_2\alpha_1 - K_1\beta_{21})}}{2(\gamma_1\alpha_2 - \gamma_2\beta_{12})}$, $B_1 = K_1\gamma_2 - k_2\gamma_1 + \beta_{12}\beta_{21} - \alpha_1\alpha_2$, $B_2 = K_2\gamma_1 - K_1\gamma_2 + \beta_{12}\beta_{21} - \alpha_1\alpha_2$	$\frac{\beta_{12}}{\alpha_2} < \frac{K_1}{K_2} < \frac{\alpha_1}{\beta_{21}}$, $\frac{\beta_{12}}{\alpha_2} < \frac{\gamma_1}{\gamma_2} < \frac{\alpha_1}{\beta_{21}}$	whenever E^* exists

Euler discrete system

Wu and Zhang [10] discretized the system (13) by Euler forward method as follows:

$$\begin{aligned} x_{n+1} &= x_n + hx_n(K_1 - \alpha_1 x_n - \beta_{12} y_n - \gamma_1 x_n y_n), \\ y_{n+1} &= y_n + hy_n(K_2 - \alpha_2 y_n - \beta_{21} x_n - \gamma_2 x_n y_n), \end{aligned} \tag{14}$$

where h is the step-size.

- It is shown that E^* may be sink or source or saddle or non-hyperbolic depending on the step-size.
- It is also shown that the system may exhibit chaos through period doubling bifurcation when step-size is varied.

10. Wu, D. and Zhang, H. (2014). Bifurcation analysis of a two-species competitive discrete model of plankton allelopathy; Advance Diff. Equ. 70.

We employ the following non-local approximations termwise for the system (13):

$$\begin{cases} \frac{dx}{dt} \to \frac{x_{n+1}-x_n}{h}, & \frac{dy}{dt} \to \frac{y_{n+1}-y_n}{h}, \\ x \to x_n, & y \to y_n, \\ xy \to y_n x_{n+1}, & xy \to x_{n+1} y_{n+1}, \\ x^2 \to x_n x_{n+1}, & y^2 \to y_n y_{n+1}, \\ x^2 y \to y_n x_n x_{n+1}, & xy^2 \to x_{n+1} y_n y_{n+1}, \end{cases} \tag{15}$$

where h (> 0) is the step-size. We get the following NSFD system.

$$\begin{aligned} x_{n+1} &= F(x_n, y_n), \\ y_{n+1} &= G(x_{n+1}, y_n), \end{aligned} \tag{16}$$

where, $F(x, y) = \frac{(1+hK_1)x}{[1+h\alpha_1 x+h\beta_{12} y+h\gamma_1 xy]}$, $G(x, y) = \frac{(1+hK_2)y}{[1+h\alpha_2 y+h\beta_{21} x+h\gamma_2 xy]}$.

Some results of NSFD system (16)

Theorem

(i) *The fixed point* $E_0 = (0,0)$ *is always a source.*

(ii) *The fixed point* $E_1 = (\frac{K_1}{\alpha_1}, 0)$ *is stable if* $\frac{\alpha_1}{\beta_{21}} < \frac{K_1}{K_2}$ *and it can not be a source. It is a saddle point if* $\frac{\alpha_1}{\beta_{21}} > \frac{K_1}{K_2}$.

(iii) *The fixed point* $E_2 = (0, \frac{K_2}{\alpha_2})$ *is stable if* $\frac{K_1}{K_2} < \frac{\beta_{12}}{\alpha_2}$ *and it can not be a source. It is a saddle point if* $\frac{K_1}{K_2} > \frac{\beta_{12}}{\alpha_2}$.

(iv) *The coexistence fixed point* E^* *is stable whenever it exists.*

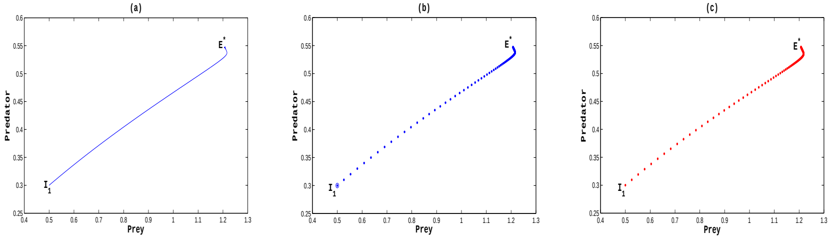
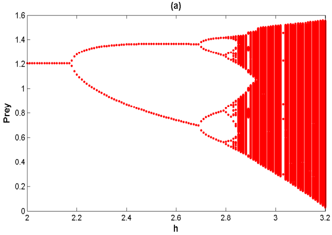


Figure 3.1. Figures (a)-(c) are, respectively, the phase portraits of the continuous system (13), Euler system (14) and NSFD system (16). Parameters are (Wu and Zhang [10]) $\alpha_1 = 0.5$, $\alpha_2 = 0.4$, $\beta_{12} = 0.3$, $\beta_{21} = 0.25$, $K_1 = 0.9$, $K_2 = 0.6$, $\gamma_1 = 0.2$, $\gamma_2 = 0.12$ and the initial point is $I_1(x_0, y_0) = (0.5, 0.3)$. The step-size is $h = 0.01$ for two discrete systems.



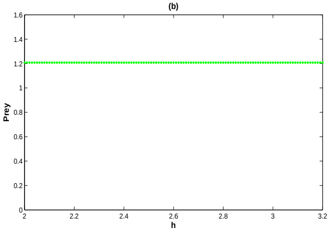


Figure 3.2. Bifurcation diagrams of prey population of the Euler system (Fig. (a)) and NSFD system (Fig. (b)) with step-size *h* as the bifurcation parameter. Parameters and initial point are as in Fig. 3.1.

4. Dynamic consistency of an eco-epidemiological model

We consider the following model studied by Bairagi et al. (2007) [11].

$$\begin{aligned} \frac{dS}{dT} &= rS(1-\frac{S+I}{K})-\lambda SI-\alpha SP, \quad S(0)=S_0>0, \\ \frac{dI}{dT} &= \lambda SI-\beta IP-\mu I, \quad I(0)=I_0\geq 0, \\ \frac{dP}{dT} &= -\theta\beta IP-\delta P+\theta\alpha SP, \quad P(0)=P_0>0. \end{aligned} \tag{17}$$

With the transformations $s=\frac{S}{K}, i=\frac{I}{K}, p=\frac{P}{K}, t=\lambda KT$, (17) becomes

$$\begin{aligned} \frac{ds}{dt} &= bs(1-(s+i))-si-m_1sp, \quad s(0)=s_0>0, \\ \frac{di}{dt} &= si-dip-e_1i, \quad i(0)=i_0\geq 0, \\ \frac{dp}{dt} &= -\theta dip-gp+\theta m_1sp, \quad p(0)=p_0>0, \end{aligned} \tag{18}$$

where $b=\frac{r}{\lambda K}$, $m_1=\frac{\alpha}{\lambda}$, $d=\frac{\beta}{\lambda}$, $e_1=\frac{\mu}{\lambda K}$ and $g=\frac{\delta}{\lambda K}$.

11. Bairagi, et al. (2007) J. Theo. Biol. 248: 10-25.

Equilibrium points of the continuous-time system (18)

The continuous system (18) has four boundary equilibrium points and one interior equilibrium point:

- $E_0 = (0, 0, 0)$ always exists.
- $E_1 = (1, 0, 0)$ always exists.
- $E_2 = \left(e_1, \frac{b(1-e_1)}{b+1}, 0\right)$ exists if $e_1 < 1$.
- $E_3 = \left(\frac{g}{\theta m_1}, 0, \frac{b(\theta m_1 - g)}{\theta m_1^2}\right)$ exists if $m_1 > \frac{g}{\theta}$.
- $E^* = (s^*, i^*, p^*)$, where, $s^* = \frac{bd\theta + m_1 e_1 \theta + bg + g}{\theta(bd + bm_1 + 2m_1)}$, $i^* = \frac{m_1 \theta s^* - g}{d\theta}$, $p^* = \frac{s^* - e}{d}$, exists if $e_1 < 1$, $m_1 > \frac{g}{\theta}$ and $\max\{e_1, \frac{g}{m_1 \theta}\} < s^* < \frac{g + d\theta}{\theta(m_1 + d)}$.

Stability of equilibrium points of the continuous system (18) (Bairagi et al. [11]).

Theorem 3.1

(i) The trivial equilibrium is always unstable.

(ii) The equilibrium E_1 is locally asymptotically stable if $e_1 > 1$ and $m_1 < \frac{g}{\theta}$.

(iii) The predator-free equilibrium E_2 is locally asymptotically stable if $e_1 < 1$ and $m_1 < \frac{1}{e_1\theta}[g + \frac{bd\theta(1-e_1)}{b+1}]$.

(iv) The infection-free equilibrium E_3 is locally asymptotically stable if $m_1 > \frac{g}{e\theta}$ or $m_1 > \frac{g}{\theta}$ according as $e_1 < 1$ or $e_1 > 1$.

(v) The interior equilibrium E^* is always unstable for all parametric values.

11. Bairagi, et al. (2007) J. Theo. Biol. 248: 10-25.

Euler discrete system

Using Euler's forward method, the continuous-time system (18) can be discretized as

$$\begin{aligned} s_{n+1} &= s_n + h[bs_n(1-(s_n+i_n)) - s_n i_n - m_1 s_n p_n], \\ i_{n+1} &= i_n + h[s_n i_n - d i_n p_n - e_1 i_n], \\ p_{n+1} &= p_n + h[-\theta d i_n p_n - g p_n + \theta m_1 s_n p_n], \end{aligned} \tag{19}$$

where $h > 0$ is the step–size.

Note

Right hand side of (19) contains many negative signs and therefore there is no guarantee of positivity of solutions for all step–size h even when the initial values are assumed to be positive. Thus, there is a possibility of numerical instabilities.

Jury conditions for a 3rd degree polynomial

Jury's Lemma: Suppose the characteristics polynomial $p(\lambda)$ is given by

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

The roots $\lambda_i, i = 1, 2, 3$, of $p(\lambda) = 0$ satisfy $|\lambda_i| < 1$ iff

(i) $p(1) = 1 + a_1 + a_2 + a_3 > 0$,

(ii) $(-1)^3 p(-1) = 1 - a_1 + a_2 - a_3 > 0$,

(iii) $1 - (a_3)^2 > |a_2 - a_3 a_1|$.

Theorem 3.2

(i) The fixed point $E_0(0,0,0)$ is always unstable.

(ii) The fixed point $E_1(1,0,0)$ is stable if $e_1 > 1$, $m_1 < \frac{g}{\theta}$ and $h < min\{\frac{2}{b}, \frac{2}{e_1-1}, \frac{2}{g-\theta m_1}\}$.

(iii) System (19) is locally asymptotically stable around $E_2\left(e_1, \frac{b(1-e_1)}{b+1}, 0\right)$ if $e_1 < 1$, $m_1 < \frac{1}{e_1\theta}[g + \frac{bd\theta(1-e_1)}{b+1}]$ and $h < min\left\{\frac{2}{(g-m_1\theta e_1)+\frac{bd\theta(1-e_1)}{b+1}}, \frac{1}{1-e_1}, \frac{2}{be_1}\right\}$.

(iv) System (19) is locally asymptotically stable around $E_3\left(\frac{g}{\theta m_1}, 0, \frac{b}{\theta m_1^2}(\theta m_1 - g)\right)$ if $m_1 > \frac{g}{e_1\theta}$ or $m_1 > \frac{g}{\theta}$ according as $e_1 < 1$ or $e_1 > 1$ and $h < min\left\{\frac{2}{(e_1-\frac{g}{\theta m_1})+\frac{bd}{\theta m_1^2}(\theta m_1-g)}, \frac{2\theta m_1}{bg}, \frac{1}{\theta m_1-g}\right\}$.

(v) System (19) is always unstable around E^* for all parameter values.

NSFD discrete system

We employ the following non-local approximations term-wise to the continuous-time system (18):

$$\begin{cases} \frac{ds}{dt} \to \frac{s_{n+1}-s_n}{h}, & \frac{di}{dt} \to \frac{i_{n+1}-i_n}{h}, & \frac{dp}{dt} \to \frac{p_{n+1}-p_n}{h}, \\ s \to s_n, & si \to s_{n+1} i_n, & ip \to i_n p_{n+1}, \\ s^2 \to s_n s_{n+1}, & ip \to i_{n+1} p_n, & p \to p_{n+1}, \\ si \to s_{n+1} i_n, & i \to i_{n+1}, & sp \to s_{n+1} p_n, \\ sp \to s_{n+1} p_n, & & \end{cases}$$

We get the following NSFD system

$$\begin{aligned} s_{n+1} &= \frac{[1+bh]s_n}{[1+h(bs_n+bi_n+i_n+m_1 p_n)]}, \\ i_{n+1} &= \frac{[1+hs_n]i_n}{[1+h(e_1+dp_n)]}, \\ p_{n+1} &= \frac{[1+h\theta m_1 s_n]p_n}{[1+h(g+\theta d i_n)]}. \end{aligned} \tag{20}$$

Theorem 3.3

(i) The fixed point $E_0 = (0, 0, 0)$ is always unstable.

(ii) The fixed point $E_1 = (1, 0, 0)$ is stable if $e_1 > 1$ and $m_1 < \frac{g}{\theta}$.

(iii) System (20) is locally asymptotically stable around $E_2 = (e_1, \frac{b(1-e_1)}{b+1}, 0)$ if $e_1 < 1$ and $m_1 < \frac{1}{e_1\theta}[g + \frac{bd\theta(1-e_1)}{b+1}]$.

(iv) System (20) is locally asymptotically stable around $E_3 = (\frac{g}{\theta m_1}, 0, \frac{b(\theta m_1 - g)}{\theta m_1^2})$ if $m_1 > \frac{g}{e_1\theta}$ or $m_1 > \frac{g}{\theta}$ according as $e_1 < 1$ or $e_1 > 1$.

(v) System (20) is always unstable around E^* for all parameter values.

Note

Note that the the dynamic properties of the NSFD system are identical with that of the continuous system.

Parameter definition and their values (Bairagi et al. [11])

Para.	Description	Unit	Value
r	Intrinsic growth rate of susceptible prey	Per day	3
K	Carrying capacity	Number per unit designated area	45
a	Half-saturation constant	Number per unit designated area	15
λ	Force of infection	Per day	0.005
α	Attack rate on susceptible prey	Per day	Varies
β	Attack rate on infected prey	Per day	0.05
μ	Death rate of infected prey due to all causes except predation	Per day	Varies
θ	Conversion efficiency	Per day	0.4
δ	Natural death rate of predator	Per day	0.09

11. Bairagi, N. et al. (2007). Role of infection on the stability of a predator-prey system with several response functions - A comparative study; J. Theo. Biol. 248: 10-25.

Comparison of dynamic results

Table 2. Stability and instability of different fixed points

fixed points	Continuous system	Euler discrete System	NSDF System	Considered m_1 & e_1 for all systems
E_0	Unstable	Unstable	Unstable	For all values
E_1	LAS if $e_1 > 1$ $m_1 < \frac{g}{\theta}$	LAS if $e_1 > 1$ $m_1 < \frac{g}{\theta}(= 1.0)$ $h(= 0.13) < min\{0.15, 30, 12.50\}$	Same with continuous system	$m_1 = 0.6$, $e_1 = 1.0667$
E_2	LAS if $e_1 < 1$ $m_1 <$ $\frac{1}{e_1 \theta}\left(g + \frac{bd\theta(1-e_1)}{b+1}\right)$	LAS if $e_1 < 1$ $m_1 < \frac{1}{e_1 \theta}\left(g + \frac{bd\theta(1-e_1)}{b+1}\right) = 23.8123$ $h < min\{\frac{2}{(g-m_1\theta e_1)+\frac{bd\theta(1-e_1)}{b+1}}, \frac{1}{1-e_1}, \frac{2}{be_1}\}$ i.e., $h(= 0.1) < min\{0.6801, 1.4516, 0.4821\}$	Same with continuous system	$m_1 = 0.18$, $e_1 = 0.3111$
E_3	LAS if $m_1 > \frac{g}{\theta}$, $e_1 > 1$	LAS if $m_1 > \frac{g}{\theta}(= 1.0)$ $h < min\{\frac{2}{(e_1 - \frac{g}{\theta m_1})+\frac{bd(\theta m_1 - g)}{\theta m_1^2}}, \frac{2\theta m_1}{bg}, \frac{1}{\theta m_1 - g}\}$ i.e., $h(= 0.063) < min\{0.0631, 0.24, 4.1667\}$	Same with continuous system	$m_1 = 1.6$, $e_1 = 1.0667$
E^*	Always unstable	Always unstable	Always unstable	For all values

Simulation results for disease-free and predator-free fixed point E_1.

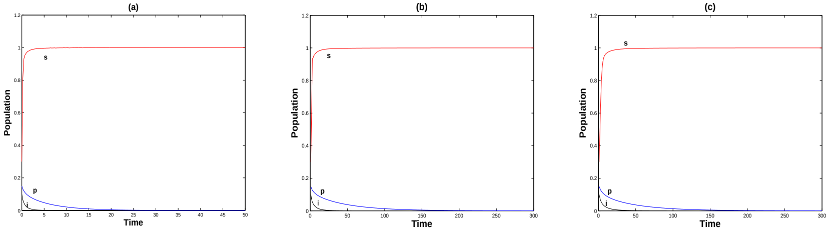


Figure 4.1. Time evolutions of three systems: (a) Continuous-time system (18), (b) Euler system (19) and (c) NSFD system (20). Initial point is taken as $I_1 = (0.30, 0.10, 0.15)$ and the step-size is taken as $h = 0.13$ for Figs. (b) and (c). Parameters are as in Table 2 with $\alpha = 0.003$ and $\mu = 0.24$. These figures show that the disease-free and predator-free fixed point is stable in all three systems when step-size is small.

Effect of step-size on susceptible prey of E_1

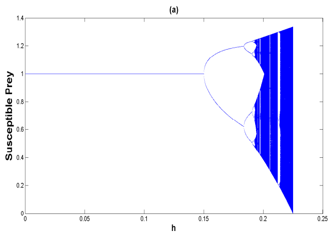


Figure 4.2. (a) Bifurcation diagrams of susceptible prey in Euler system (Fig. a) and that of NSFD system (Fig. b) with step-size h as the bifurcation parameter. Initial value and other parameters are as in Fig. 4.1.

Simulation results for predator-free fixed point E_2.

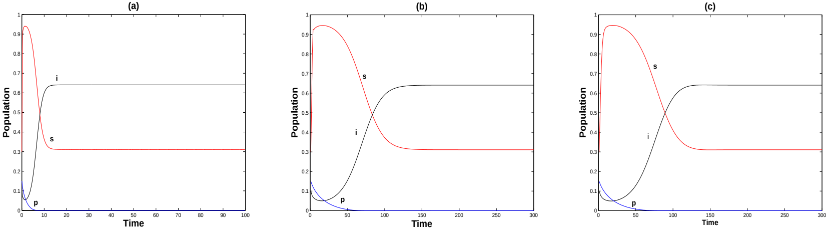


Figure 4.3. Figs. (a), (b) and (c) represent, respectively, the time series solutions of systems (18), (19) and (20). In Figs. (b) and (c), step-size is taken as $h = 0.1$. These figures show that the fixed point E_2 is stable in all three cases when step-size is sufficiently small. Parameter values and initial point are as in Fig. 4.1 except $\alpha = 0.0009$ and $\mu = 0.07$..

Effect of step-size on susceptible prey of E_2

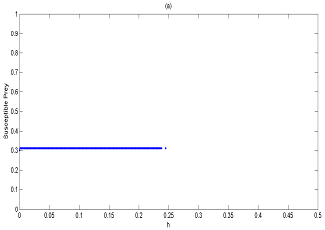


Figure 4.4. Bifurcation diagram of susceptible prey population of Euler model (19) (Fig. a) and that of NSFD model (20) (Fig. b) with variable step-size. Parameters and initial value are as in Fig. 4.3. Figure (a) shows that the prey population is stable for step-size up to 0.23 and integration failure occurs for higher values of h. The second figure shows that the prey population is stable for all step-size h.

Simulation results for infection-free fixed point E_3.

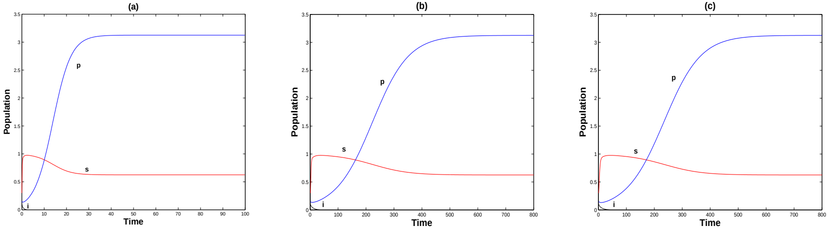


Figure 4.5. Time evolutions of systems (18), (19) and (20) for $e_1 > 1$. Each system reaches to disease-free fixed point E_3 when step-size is small (h=0.063). Other parameter are as in Fig. 4.1.

Effect of step-size on susceptible prey of E_3

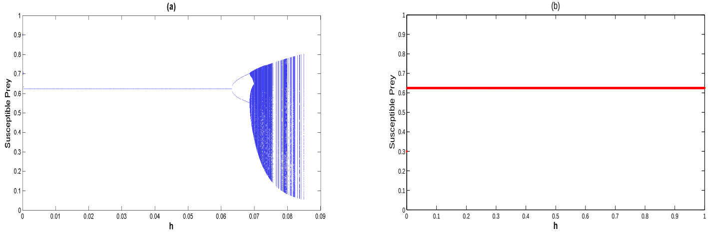


Figure 4.6. (a) Bifurcation diagram of susceptible prey population of Euler model (19). It shows that the prey population is stable for small step–size h and unstable for higher values of h. (b) Bifurcation diagram of NSFD model (20) shows that the susceptible prey population is stable for all step–size h. Parameter values and initial point are same as in Fig. 4.5.

Summary

- Standard finite difference (SFD) method can not preserves two important dynamical properties, namely stability of the equilibria and positivity of the solutions for any step-size.
- SFD method also shows some spurious behaviors like chaos which may not the properties of original continuous system and therefore dynamically inconsistent.
- We have shown that solution of the NSFD systems remain positive for all positive initial values.
- Our proposed NSFD systems preserve the stability of fixed points for arbitrary step-size and therefore dynamically consistent.

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Thank You