

Some conditions of eradication of epidemic process
within stochastic and deterministic SIRS models

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1. Statement of the problem.

Suppose that we deal with individuals living in some region.

The community of individuals of the region is splitted into three classes: susceptible (S), infected (I), resistant (R).

Transitions of individuals between classes define model name:

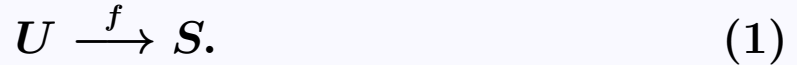
$$S \longrightarrow I \longrightarrow R \longrightarrow S.$$

Denote by U — an auxiliary class, that contains individuals of other regions and died individuals.

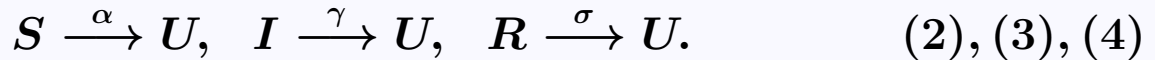
PURPOSE: the comparison of conditions for the eradication of the epidemic process within the stochastic and deterministic models depending on the Basic reproduction number R_0 .

2. Model assumptions.

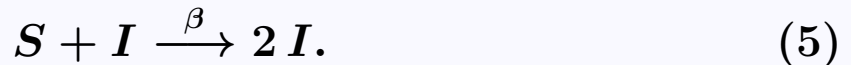
- Class S can be replenished by individuals from other regions:



- Each individual can die or migrate to other regions:



- Contact of susceptible and infected individuals leads to infection of the sensitive individual:



- The infected individual may recover resistance to infection. Eventually the resistance is lost and individual becomes susceptible to infection:



Here: $f, \alpha, \gamma, \sigma, \beta$ — positive constants, F_I, F_R — distribution functions that specify the time of stay of an individual in the corresponding class.

- Suppose, that F_I and F_R are exponential distribution functions ($\lambda = \text{const} > 0$, $\mu = \text{const} > 0$):

$$F_I(a) = 1 - e^{-\lambda a}, \quad F_R(a) = 1 - e^{-\mu a}, \quad a \geq 0. \quad (8)$$

In this case, a deterministic model — the system of ordinary differential equations,

stochastic model — a non-linear birth and death process.

- Suppose, that F_I or F_R differs from functions (8).

Then we have to take into account the history of the development of epidemic process:

if we fix time t , then we must use information about the process at previous time $s \leq t$.

Next, we will present our results for one partial case, namely:

$$F_I(a) = 0, a \leq \omega_I, \quad F_I(a) = 1, a > \omega_I, \quad (9)$$

$$F_R(a) = 0, a \leq \omega_R, \quad F_R(a) = 1, a > \omega_R, \quad (10)$$

where $\omega_I = \text{const} > 0$, $\omega_R = \text{const} > 0$ — length of stay of individuals in classes I and R .

- Deterministic model—the system of delay differential equations.
- Stochastic model—Markov random process in a special state space.

3. Model in the form of delay differential equations.

Denote by $x(t)$, $y(t)$, $z(t)$ the numbers of individuals of classes S , I , R at time t (real variables). Model equations:

$$dx(t)/dt = f - \alpha x(t) - \beta x(t)y(t) + r_2 \beta x(t - \omega_I - \omega_R)y(t - \omega_I - \omega_R),$$

$$dy(t)/dt = \beta x(t)y(t) - \gamma y(t) - r_1 \beta x(t - \omega_I)y(t - \omega_I),$$

$$dz(t)/dt = r_1 \beta x(t - \omega_I)y(t - \omega_I) - \sigma z(t) - r_2 \beta x(t - \omega_I - \omega_R)y(t - \omega_I - \omega_R),$$

$$t \geq \omega_I + \omega_R, \quad (11)$$

$$x(t) = x_0(t), y(t) = y_0(t), t \in [0; \omega_I + \omega_R], z(\omega_I + \omega_R) = z_0, \quad (12)$$

$x_0(t) \geq 0$, $y_0(t) \geq 0$ – continuous functions, $z_0 = \text{const} \geq 0$,

$$r_1 = \exp(-\gamma\omega_I), r_2 = \exp(-\gamma\omega_I - \sigma\omega_R).$$

The initial data (12) may be specified by means of auxiliary system of differential equations.

In this case the system (11), (12) has on the interval $[\omega_I + \omega_R; \infty)$ the unique solution with nonnegative components.

Denote (the Basic reproduction number)

$$R_0 = \frac{f \beta (1 - r_1)}{\alpha \gamma}. \quad (13)$$

The system (11) for any values of parameters has a trivial equilibrium

$$x_1^* = \frac{f}{\alpha} > 0, \quad y_1^* = 0, \quad z_1^* = 0. \quad (14)$$

If $R_0 < 1$, then the system (11) has no more equilibria with nonnegative components, except (14), and (14) is asymptotically stable.

If $R_0 > 1$, then the equilibrium (14) is not stable, and the system (11) has one more equilibrium with positive components:

$$\begin{aligned}x_2^* &= \frac{\gamma}{\beta(1 - r_1)} > 0, & y_2^* &= \frac{\alpha(R_0 - 1)}{\beta(1 - r_2)} > 0, \\z_2^* &= \frac{(r_1 - r_2)\beta x_2^* y_2^*}{\sigma} > 0.\end{aligned}\tag{15}$$

We do not have general results regarding the stability or instability of the equilibrium (15), except some partial cases.

4. Stochastic SIRS model.

Let $x(t)$, $y(t)$, $z(t)$ — random integer variables — the numbers of individuals of classes S , I , R at time t . To distinguish individuals of the classes I and R we will use two special sets:

$$\Omega_I(t) = \{a_{I1} + \omega_I, \dots, a_{Ik} + \omega_I, \dots, a_{Iy(t)} + \omega_I\}, \quad (16)$$

$$\Omega_R(t) = \{b_{R1} + \omega_R, \dots, b_{Rj} + \omega_R, \dots, b_{Rz(t)} + \omega_R\}, \quad (17)$$

$$\begin{aligned} a_{I1} < \dots < a_{Iy(t)} \leq t, & \quad a_{Ik} + \omega_I > t, \quad 1 \leq k \leq y(t), \\ b_{R1} < \dots < b_{Rz(t)} \leq t, & \quad b_{Rj} + \omega_R > t, \quad 1 \leq j \leq z(t). \end{aligned}$$

In (16) the symbol a_{Ik} means time moment when the individual of class S entered into class I ; the symbol $a_{Ik} + \omega_I$ means the time of possible transition of an individual from class I to class R . In (17) the symbols b_{Rj} , $b_{Rj} + \omega_R$ have the same meaning.

If $y(t) = 0$ or $z(t) = 0$, we put $\Omega_I(t) = \emptyset$ or $\Omega_R(t) = \emptyset$.

To describe the dynamics of $x(t)$, $y(t)$, $z(t)$, $\Omega_I(t)$, $\Omega_R(t)$ we use the following approach.

Let t , $x(t)$, $y(t)$, $z(t)$, $\Omega_I(t)$, $\Omega_R(t)$ are fixed. We accept:

$$\begin{aligned}\tau &= \min\{a_{I1} + \omega_I, b_{R1} + \omega_R, t + \xi_1, \dots, t + \xi_5\}, \\ x(\tau) &= x(t) + \Delta x, \quad y(\tau) = y(t) + \Delta y, \quad z(\tau) = z(t) + \Delta z, \\ \Omega_I(\tau) &= \Omega_I(t) \pm \{a_{In}\}, \quad \Omega_R(\tau) = \Omega_R(t) \pm \{b_{Rn}\}.\end{aligned}$$

Symbol \pm means replenishment of $\Omega_I(t)$, $\Omega_R(t)$ by new elements or an exception of some elements from $\Omega_I(t)$, $\Omega_R(t)$.

Elements $a_{I1} + \omega_I$, $b_{R1} + \omega_R$ are the first one of the $\Omega_I(t)$, $\Omega_R(t)$.

Distributions of random variables

$$\xi_1, \dots, \xi_5, \Delta x, \Delta y, \Delta z, a_{In}, b_{Rn} \quad (18)$$

are set on the basis of transition probabilities during interval $(t, t + h)$, $h \rightarrow +0$ (in accordance with the formulas (1)–(5)):

$$\begin{aligned}
P\{(x(t), y(t), z(t)) \rightarrow (x(t) + 1, y(t), z(t))\} &= f h + o(h), \\
P\{(x(t), y(t), z(t)) \rightarrow (x(t) - 1, y(t), z(t))\} &= \alpha x(t) h + o(h), \\
P\{(x(t), y(t), z(t)) \rightarrow (x(t), y(t) - 1, z(t))\} &= \gamma y(t) h + o(h), \\
P\{(x(t), y(t), z(t)) \rightarrow (x(t), y(t), z(t) - 1)\} &= \sigma z(t) h + o(h), \\
P\{(x(t), y(t), z(t)) \rightarrow (x(t) - 1, y(t) + 1, z(t))\} &= \beta x(t) y(t) h + o(h), \\
P\{(x(t), y(t), z(t)) \rightarrow (x(t), y(t), z(t))\} &= 1 - q(x(t), y(t), z(t)) h + o(h),
\end{aligned}$$

$$q(x(t), y(t), z(t)) = q_1(x(t), y(t), z(t)) + \dots + q_5(x(t), y(t), z(t)),$$

$$q_1(x(t), y(t), z(t)) = f; \quad q_2(x(t), y(t), z(t)) = \alpha x(t); \quad q_3(x(t), y(t), z(t)) = \gamma y(t);$$

$$q_4(x(t), y(t), z(t)) = \sigma z(t); \quad q_5(x(t), y(t), z(t)) = \beta x(t) y(t).$$

For example, if $x(t) \neq 0$, $\Omega_I(t) \neq \emptyset$, $\Omega_R(t) \neq \emptyset$, then:

$$P\{\xi_i < a\} = 1 - e^{-q_i(x(t), y(t), z(t)) a}, \quad a \geq 0, \quad 1 \leq i \leq 5. \quad (19)$$

The distribution laws of random variables Δx , Δy , Δz and a_{In} , b_{Rn} depend on variable τ .

5. Results of numerical simulations.

The purpose of numerical simulation — studying the eradication of the infection in the stochastic model at fixed time T and fixed initial state \mathcal{F}_0 .

We study this problem depending on basic reproduction number R_0 , defined by formula (13).

We used parameters and initial data:

$$\begin{array}{lll} f = 10000.0, & \alpha = 1.0, & x(0) = 8000, \\ \beta = 0.002, & \gamma = 2.0, & y(0) = 1000, \\ \omega_R = 10 \omega_I, & \sigma = 1.0, & z(0) = 1000, \end{array}$$

The value of the parameter $\alpha = 1.0$ is chosen such, that the average lifespan of individuals of the class S equals $1/\alpha = 1.0$.

Parameter ω_I takes different values.

For the numerical simulation we used own computer program Populations Modeler, developed since 2004 year.

To make estimations, we calculated and averaged $n = 10000$ realizations of the stochastic process.

- Let us introduce the probability of eradication of an infectious process at time $T = 20$:

$$P_D = \mathbf{P}(y(T) = 0 \mid \mathcal{F}_0).$$

For the above model parameters the estimation error of \bar{P}_D for P_D does not exceed 0.0131 at 99%–confidence level.

- For expectation $\mathbf{E}(y(T) \mid \mathcal{F}_0)$ the estimation error of $\bar{y}(T)$ does not exceed 5.56 at 99%–confidence level.

The calculation results are shown in table

R_0	ω_I	y_2^*	\bar{P}_D
0.90	0.0958	–	1.0
1.00	0.1073	–	1.0
1.09	0.1177	90.1	0.96
1.15	0.1248	144.4	0.42
1.21	0.1319	194.9	0.12
1.30	0.1427	264.8	0.05
1.40	0.1548	335.9	0.0
1.50	0.1672	401.4	0.0

One can see, that in the cases $1 < R_0 < 1.4$ the probability of eradication of the epidemic process $P_D > 0$.

But, for deterministic model it is NOT impossible, that

$$y(t) \rightarrow 0, \quad t \rightarrow +\infty.$$

Let us consider the calculation results in the case $R_0 = 1.21$.

We have received $\bar{y}(T) = 172.3$.

As a consequence, $\mathbf{E}(y(T) \mid \mathcal{F}_0) < y_2^* = 194.9$.

Nevertheless, $\mathbf{E}(y(T) \mid \mathcal{F}_0) \approx y_2^*(1 - P_D)$,

and conditional expectation $\mathbf{E}(y(T) \mid \mathcal{F}_0, y(T) > 0) \approx y_2^* !!!$

6. Conclusion.

Conditions of eradication of an infection in the case $R_0 > 1$ are different in the framework of stochastic and deterministic models. Main reasons:

- 1) the structure of the used models, as well as of continuity and discontinuity models variables;
- 2) unstability of equilibrium $(x_1^*, y_1^*, z_1^*) = (f/\alpha, 0, 0)$ in deterministic model;
- 3) the presence of absorb state — point $y = 0$ in stochastic model;
- 4) a small number of individuals of class I .

Therefore, the results of the research of deterministic models should be supplemented by the results of numerical experiments with the stochastic models.

Thank You For Attention !

Supplement 1.

The stochastic SIRS-model allows such mode at which $y(t) = 0, z(t) = 0$ almost surely for each fixed $t \geq 0$.

In this case the random process $x(t)$ is the linear random death process with immigration of particles, and

$$\mathbf{E}(x(t) \mid \mathcal{F}_0) = (x(0) - f/\alpha) e^{-\alpha t} + f/\alpha, \quad t \geq 0.$$

To be able to perform numerical experiments it is important to show that

$$\mathbf{E}(w^2(t) \mid \mathcal{F}_0) < +\infty, \quad t \geq 0,$$

where $w(t) = x(t) + y(t) + z(t)$,

$$\mathcal{F}_0 = \{x(t_0), y(t_0), z(t_0), \Omega_I(t_0), \Omega_R(t_0)\}.$$

We have proved that

$$\mathbf{E}(w^2(t) \mid \mathcal{F}_0) \leq w^2(0) + (1 + 2w(0))ft + (ft)^2 < +\infty, \quad t \geq 0.$$

Supplement 2.

The program Populations Modeler uses:

- the Monte–Carlo technique;
- 128-bit congruent pseudorandom number generator;
- the big frog and small frog technique to select random numbers subsequents for realizations;
- an effective memory model allowing to process about 10^6 – 10^7 individuals;
- 128-bit representation of time points to prevent rounding errors;
- multi-thread and cluster computations.