Some conditions of eradication of epidemic process within stochastic and deterministic SIRS models

N. Pertsev and B. Pichugin

Sobolev Institute of Mathematics,
Omsk Division

BIOMAT–2017

Moscow

1. Statement of the problem.

Suppose that we deal with individuals living in some region.

The community of individuals of the region is splitted into three classes: susceptible (S), infected (I), resistant (R).

Transitions of individuals between classes define model name:

$$S \longrightarrow I \longrightarrow R \longrightarrow S.$$

Denote by U — an auxiliary class, that contains individuals of other regions and died individuals.

PURPOSE: the comparison of conditions for the eradication of the epidemic process within the stochastic and deterministic models depending on the Basic reproduction numder R_0.

2. Model assumptions.

- Class S can be replenished by individulas from other regions:

$$U \xrightarrow{f} S. \tag{1}$$

- Each individual can die or migrate to other regions:

$$S \xrightarrow{\alpha} U, \quad I \xrightarrow{\gamma} U, \quad R \xrightarrow{\sigma} U. \qquad (2), (3), (4)$$

- Contact of susceptible and infected individuals leads to infection of the sensitive individual:

$$S + I \xrightarrow{\beta} 2\, I. \tag{5}$$

- The infected individual may recover resistance to infection. Eventually the resistance losts and individual becomes susceptible to infection:

$$I \overset{F_I}{\Longrightarrow} R, \quad R \overset{F_R}{\Longrightarrow} S. \qquad (6), (7)$$

Here: f, α, γ, σ, β — positive constants, F_I, F_R — distribution functions that specify the time of stay of an individual in the corresponding class.

• Suppose, that F_I and F_R are exponential distribution functions ($\lambda = const > 0, \quad \mu = const > 0$):

$$F_I(a) = 1 - e^{-\lambda a}, \quad F_R(a) = 1 - e^{-\mu a}, \quad a \geq 0. \tag{8}$$

In this case, a deterministic model — the system of ordinary differential equations,

stochastic model — a non-linear birth and death process.

• Suppose, that F_I or F_R differs from functions (8).

Then we have to take into account the history of the development of epidemic process:

if we fix time t, then we must use information about the process at previous time $s \leq t$.

Next, we will present our results for one partial case, namely:

$$F_I(a) = 0,\ a \leq \omega_I, \quad F_I(a) = 1,\ a > \omega_I, \tag{9}$$

$$F_R(a) = 0,\ a \leqslant \omega_R, \quad F_R(a) = 1,\ a > \omega_R, \tag{10}$$

where $\omega_I = const > 0$, $\omega_R = const > 0$ — length of stay of individuals in classes I and R.

• Deterministic model—the system of delay differential equations.

• Stochastic model—Markov random process in a special state space.

3. Model in the form of delay differential equations.

Denote by $x(t)$, $y(t)$, $z(t)$ the numbers of individuals of classes S, I, R at time t (real variables). Model equations:

$$dx(t)/dt = f - \alpha x(t) - \beta x(t)y(t) + r_2\beta x(t-\omega_I-\omega_R)y(t-\omega_I-\omega_R),$$

$$dy(t)/dt = \beta x(t)y(t) - \gamma y(t) - r_1\beta x(t-\omega_I)y(t-\omega_I),$$

$$dz(t)/dt = r_1\beta x(t-\omega_I)y(t-\omega_I) - \sigma z(t) -$$
$$- r_2\beta x(t-\omega_I-\omega_R)y(t-\omega_I-\omega_R),$$
$$t \geq \omega_I + \omega_R, \tag{11}$$

$$x(t) = x_0(t), y(t) = y_0(t), t \in [0; \omega_I + \omega_R], z(\omega_I + \omega_R) = z_0, \tag{12}$$

$x_0(t) \geq 0$, $y_0(t) \geq 0$ — continuous functions, $z_0 = const \geqslant 0$,

$$r_1 = \exp(-\gamma\omega_I),\ r_2 = \exp(-\gamma\omega_I - \sigma\omega_R).$$

The initial data (12) may be specified by means of auxiliary system of differential equations.
In this case the system (11), (12) has on the interval $[\omega_I+\omega_R;\infty)$ the unique solution with nonnegative components.

Denote (the Basic reproduction numder)

$$R_0 = \frac{f\,\beta\,(1-r_1)}{\alpha\,\gamma}. \tag{13}$$

The system (11) for any values of parameters has a trivial equilibrium

$$x_1^* = \frac{f}{\alpha} > 0, \quad y_1^* = 0, \quad z_1^* = 0. \tag{14}$$

If $R_0 < 1$, then the system (11) has no more equilibriums with nonnegative components, except (14), and (14) is asymptotically stable.

If $R_0 > 1$, then the equilibrium (14) is not stable, and the system (11) has one more equilibrium with positive components:

$$x_2^* = \frac{\gamma}{\beta(1 - r_1)} > 0, \quad y_2^* = \frac{\alpha\,(R_0 - 1)}{\beta\,(1 - r_2)} > 0,$$

$$z_2^* = \frac{(r_1 - r_2)\beta\, x_2^*\, y_2^*}{\sigma} > 0. \tag{15}$$

We do not have general results regarding the stability or instability of the equilibrium (15), except some partial cases.

4. Stochastic SIRS model.

Let $x(t)$, $y(t)$, $z(t)$ — random integer variables — the numbers of individuals of classes S, I, R at time t. To distinguish individuals of the classes I and R we will use two special sets:

$$\Omega_I(t) = \{a_{I1} + \omega_I, \dots, a_{Ik} + \omega_I, \dots, a_{Iy(t)} + \omega_I\}, \qquad (16)$$

$$\Omega_R(t) = \{b_{R1} + \omega_R, \dots, b_{Rj} + \omega_R, \dots, b_{Rz(t)} + \omega_R\}, \qquad (17)$$

$$a_{I1} < \cdots < a_{Iy(t)} \le t, \quad a_{Ik} + \omega_I > t, \quad 1 \le k \le y(t),$$
$$b_{R1} < \cdots < b_{Rz(t)} \le t, \quad b_{Rj} + \omega_R > t, \quad 1 \le j \le z(t).$$

In (16) the symbol a_{Ik} means time moment when the individual of class S entered into class I; the symbol $a_{Ik} + \omega_I$ means the time of possible transition of an individual from class I to class R. In (17) the symbols b_{Rj}, $b_{Rj} + \omega_R$ have the same meaning.

If $y(t) = 0$ or $z(t) = 0$, we put $\Omega_I(t) = \emptyset$ or $\Omega_R(t) = \emptyset$.

To describe the dynamics of $x(t)$, $y(t)$, $z(t)$, $\Omega_I(t)$, $\Omega_R(t)$ we use the following approach.

Let $t, x(t), y(t), z(t), \Omega_I(t), \Omega_R(t)$ are fixed. We accept:

$$\tau = \min\{a_{I1} + \omega_I,\ b_{R1} + \omega_R,\ t + \xi_1, \dots, t + \xi_5\},$$
$$x(\tau) = x(t) + \Delta x,\ y(\tau) = y(t) + \Delta y,\ z(\tau) = z(t) + \Delta z,$$
$$\Omega_I(\tau) = \Omega_I(t) \pm \{a_{In}\},\ \Omega_R(\tau) = \Omega_R(t) \pm \{b_{Rn}\}.$$

Symbol $\pm$ means replenishment of $\Omega_I(t)$, $\Omega_R(t)$ by new elements or an exception of some elements from $\Omega_I(t)$, $\Omega_R(t)$.

Elements $a_{I1}+\omega_I$, $b_{R1}+\omega_R$ are the first one of the $\Omega_I(t)$, $\Omega_R(t)$.

Distributions of random variables

$$\xi_1, \dots, \xi_5,\ \Delta x,\ \Delta y,\ \Delta z,\ a_{In},\ b_{Rn} \tag{18}$$

are set on the basis of transition probabilities during interval $(t, t + h)$, $h \to +0$ (in accordance with the formulas (1)–(5)):

$$P\{(x(t),y(t),z(t)) \to (x(t)+1,y(t),z(t))\} = f\,h + o(h),$$
$$P\{(x(t),y(t),z(t)) \to (x(t)-1,y(t),z(t))\} = \alpha\,x(t)\,h + o(h),$$
$$P\{(x(t),y(t),z(t)) \to (x(t),y(t)-1,z(t))\} = \gamma\,y(t)\,h + o(h),$$
$$P\{(x(t),y(t),z(t)) \to (x(t),y(t),z(t)-1)\} = \sigma\,z(t)\,h + o(h),$$
$$P\{(x(t),y(t),z(t)) \to (x(t)-1,y(t)+1,z(t))\} = \beta\,x(t)\,y(t)\,h + o(h),$$
$$P\{(x(t),y(t),z(t)) \to (x(t),y(t),z(t))\} = 1 - q(x(t),y(t),z(t))\,h + o(h),$$

$$q(x(t),y(t),z(t)) = q_1(x(t),y(t),z(t)) + \cdots + q_5(x(t),y(t),z(t)),$$

$$q_1(x(t),y(t),z(t)) = f;\;\; q_2(x(t),y(t),z(t)) = \alpha\,x(t);\;\; q_3(x(t),y(t),z(t)) = \gamma\,y(t);$$
$$q_4(x(t),y(t),z(t)) = \sigma\,z(t);\;\; q_5(x(t),y(t),z(t)) = \beta\,x(t)\,y(t).$$

For example, if $x(t) \neq 0$, $\Omega_I(t) \neq \emptyset$, $\Omega_R(t) \neq \emptyset$, then:

$$P\{\xi_i < a\} = 1 - e^{-q_i(x(t),y(t),z(t))\,a},\;\; a \geq 0,\;\; 1 \leq i \leq 5. \qquad (19)$$

The distribution laws of random variables Δx, Δy, Δz and a_{In}, b_{Rn} depend on variable τ.

5. Results of numerical simulations.

The purpose of numerical simulation — studying the eradication of the infection in the stochastic model at fixed time T and fixed initial state $\mathcal{F}_0$.

We study this problem depending on basic reproduction numder R_0, defined by formula (13).

We used parameters and initial data:

$$\begin{aligned} f &= 10000.0, & \alpha &= 1.0, & x(0) &= 8000, \\ \beta &= 0.002, & \gamma &= 2.0, & y(0) &= 1000, \\ \omega_R &= 10\,\omega_I, & \sigma &= 1.0, & z(0) &= 1000, \end{aligned}$$

The value of the parameter $\alpha = 1.0$ is chosen such, that the average lifespan of individuals of the class S equals $1/\alpha = 1.0$.

Parameter ω_I takes different values.

For the numerical simulation we used own computer program `Populations Modeler`, developed since 2004 year.

To make estimations, we calculated and averaged $n = 10000$ realizations of the stochastic process.

• Let us introduce the probability of eradication of an infectious process at time $T = 20$:

$$P_D = \mathsf{P}(y(T) = 0 \mid \mathcal{F}_0).$$

For the above model parameters the estimation error of $\bar{P}_D$ for P_D does not exceed 0.0131 at 99%–confidence level.

• For expectation $\mathsf{E}(y(T) \mid \mathcal{F}_0)$ the estimation error of $\bar{y}(T)$ does not exceed 5.56 at 99%–confidence level.

The calculation results are shown in table

R_0	ω_I	y_2^*	$\bar{P}_D$
0.90	0.0958	–	1.0
1.00	0.1073	–	1.0
1.09	0.1177	90.1	0.96
1.15	0.1248	144.4	0.42
1.21	0.1319	194.9	0.12
1.30	0.1427	264.8	0.05
1.40	0.1548	335.9	0.0
1.50	0.1672	401.4	0.0

One can see, that in the cases $1 < R_0 < 1.4$ the probability of eradication of the epidemic process $P_D > 0$.

But, for deterministic model it is NOT impossible, that

$$y(t) \to 0, \quad t \to +\infty.$$

Let us consider the calculation results in the case $R_0 = 1.21$.

We have received $\bar{y}(T) = 172.3$.

As a consequence, $\mathsf{E}(y(T) \mid \mathcal{F}_0) < y_2^* = 194.9$.

Nevertheless, $\mathsf{E}(y(T) \mid \mathcal{F}_0) \approx y_2^*(1 - P_D)$,

and conditional expectation $\mathsf{E}(y(T) \mid \mathcal{F}_0,\ y(T) > 0) \approx y_2^*$!!!

6. Conclusion.

Conditions of eradication of an infection in the case $R_0 > 1$ are different in the framework of stochastic and deterministic models. Main reasons:

1) the structure of the used models, as well as of continuity and discontinuity models variables;

2) unstability of equilibrium $(x_1^*, y_1^*, z_1^*) = (f/\alpha, 0, 0)$ in deterministic model;

3) the presence of absorb state — point $y = 0$ in stochastic model;

4) a small number of individuals of class I.

Therefore, the results of the research of deterministic models should be supplemented by the results of numerical experiments with the stochastic models.

Thank You For Attention !

Supplement 1.

The stochastic SIRS-model allows such mode at which $y(t) = 0$, $z(t) = 0$ almost surely for each fixed $t \geqslant 0$.

In this case the random process $x(t)$ is the linear random death process with immigration of particles, and

$$\mathsf{E}(x(t) \mid \mathcal{F}_0) = \big(x(0) - f/\alpha\big)\, e^{-\alpha t} + f/\alpha, \quad t \geqslant 0.$$

To be able to perform numerical experiments it is important to show that

$$\mathsf{E}(w^2(t) \mid \mathcal{F}_0) < +\infty, \; t \geqslant 0,$$

where $$w(t) = x(t) + y(t) + z(t),$$

$$\mathcal{F}_0 = \{x(t_0), y(t_0), z(t_0), \Omega_I(t_0), \Omega_R(t_0)\}.$$

We have proved that

$$\mathsf{E}(w^2(t) \mid \mathcal{F}_0) \leqslant w^2(0) + (1 + 2w(0))ft + (ft)^2 < +\infty, \; t \geq 0.$$

Supplement 2.

The program `Populations Modeler` uses:

- the Monte–Carlo technique;
- 128-bit congruent pseudorandom number generator;
- the big frog and small frog technique to select random numbers subsequents for realizations;
- an effective memory model allowing to process about $10^6 - 10^7$ individuals;
- 128-bit representation of time points to prevent rounding errors;
- multi-thread and cluster computations.