DELAY INDUCED OSCILLATIONS IN A DYNAMICAL MODEL FOR INFECTIOUS DISEASE

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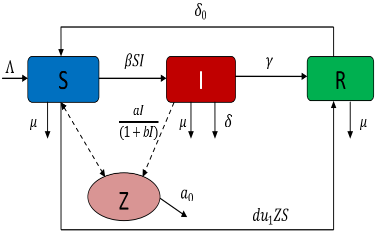
Organization of Presentation

Infectious Diseases: Impacts and Challenges

- Infectious diseases are leading cause of mortality and morbidity and have major impact on public budget.
- Infectious diseases also have serious *socioeconomic consequences* related to health care, travel and employment etc.
- Control interventions (available) improve the quality life of individuals but pose significant economic burden
- Determination and implementation of suitable control policies are still challenging. (*Prevention vs. Treatment*)
- There is a trade-off among the available control interventions (pharmaceutical and non pharmaceutical) for their implementation during the epidemic outbreak.
- Human behavioural changes due to disease presence has been found to affect disease progression.
- Delays or time lags are inherent to natural system, and so are in disease dynamics.

SIRSZ Model: Behavioural Response, Information and Treatment[1]



SIRS Model

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta SI - \mu S - f(S,Z) + \delta_0 R, \\
\frac{dI}{dt} &= \beta SI - (\mu + \delta + \gamma) I \\
\frac{dR}{dt} &= \gamma I + f(S,Z) - (\mu + \delta_0) R, \\
\frac{dZ}{dt} &= g(I) - a_0 Z,
\end{aligned}
$$

with initial conditions
$S(0) > 0,\ I(0) \geq 0, R(0) \geq 0$ and $Z(0) \geq 0$.

Functions & Parameters

- $f(S,Z) = u_1 dZS$: behavioural response, $u_1 d$: response rate.
- $0 \leq u_1 \leq 1$: response intensity.
- $g(I) = \frac{aI}{1+bI}$: growth of information, a: growth rate of information with saturation constant b.
- a_0: natural degradation rate of information.
- δ_0 $(= \delta_1 + \delta_2)$: rate of loss of total immunity.

[1] Kumar *et al.*, Journal of Theoretical Biology, 414, pp. 103-119, 2017.

Two Delay Model

Two Delay Model: Delay in information & Waning immunity

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \beta S(t)I(t) - \mu S(t) - u_1 dZ(t)S(t) + \delta_0 R(t-\tau_1), \\
\frac{dI(t)}{dt} &= \beta S(t)I(t) - (\mu+\delta+\gamma)I(t), \\
\frac{dR(t)}{dt} &= \gamma I(t) + u_1 dZ(t)S(t) - \mu R(t) - \delta_0 R(t-\tau_1), \\
\frac{dZ(t)}{dt} &= aI(t-\tau_2) - a_0 Z(t),
\end{aligned} \tag{1}$$

with initial conditions $S(\theta) = S_0 \geq 0, I(\theta) = I_0 \geq 0, R(\theta) = R_0 \geq 0$ and $Z(\theta) = Z_0 \geq 0$, $\theta \in [-max\{\tau_1, \tau_2\}, 0]$, where $(S(\theta), I(\theta), R(\theta), Z(\theta)) \in C([-max\{\tau_1, \tau_2\}, 0], R_+^4)$, the Banach space of continuous functions. Here, all parameters are non-negative.

Positivity and Boundedness

We note that all the solutions of model system (1) remain positive if we start with a positive initial function and remain ultimately bounded.
Thus the biologically feasible region of the model system (1) is the following positive invariant set:

$$\Gamma = \left\{ (S(t), I(t), R(t), Z(t)) \in \mathbb{R}^4_+ \mid 0 \leq S(t), I(t), R(t) \leq \frac{\Lambda}{\mu}, Z(t) \leq \frac{a\Lambda}{a_0 \mu} \right\}.$$

Existence of the Equilibrium Points

The basic reproduction number is given as:

$$R_0 = \frac{\Lambda\beta}{\mu(\mu+\delta+\gamma)}.$$

The model system (1) have following two equilibria:

(i) a disease free equilibrium $E_1 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ which always exists, and

(ii) a unique infected equilibrium $E_2 = (S_*, I_*, R_*, Z_*)$, which exists if and only if $R_0 > 1$.
Here $S_* = \frac{(\mu+\delta+\gamma)}{\beta}$, $R_* = \frac{I_*}{\mu+\delta_0}\left(\gamma + \frac{du_1 a(\mu+\delta+\gamma)}{a_0\beta}\right)$, $Z_* = \frac{aI_*}{a_0}$ and $I_* = \frac{C}{B}$,
where $B = \frac{\mu(\mu+\delta+\gamma)+\delta_0(\mu+\delta)}{(\mu+\delta_0)} + \frac{\mu du_1 a(\mu+\delta+\gamma)}{\beta a_0(\mu+\delta_0)}$ and $C = \Lambda\left(1 - \frac{1}{R_0}\right)$.

Stability result for $\tau_1 = \tau_2 = 0$.

Theorem

For $\tau_1 = \tau_2 = 0$,

(i) *the disease free equilibrium* E_1 *of the system (1) is locally asymptotically stable if* $R_0 < 1$ *and is unstable if* $R_0 > 1$,

(ii) *if* $R_0 > 1$ *then the unique infected equilibrium* E_2 *is locally asymptotically stable provided following conditions are satisfied:*

$$P_1 P_2 > P_3 \ \text{and} \ P_1(P_2 P_3 - P_1 P_4) > P_3^2.$$

Here $P_1 = a_0 + 2\mu + \delta_0 + \beta I_* + du_1 Z_*$,
$P_2 = a_0(\mu + \delta_0) + (\mu + a_0)(\mu + \beta I_* + du_1 Z_*) + \delta_0(\mu + \beta I_*) + \beta^2 S_* I_*$,
$P_3 = \beta I_*((a_0 + \mu)(\mu + \delta + \gamma) + \delta_0(\mu + \delta)) + \mu a(\mu + \beta I_* + du_1 Z_*) + a_0 \delta_0(\mu + \beta I_*) + adu_1 \beta S_* I_*$ *and* $P_4 = \beta a_0 I_*(\mu(\mu + \delta + \gamma) + \delta_0(\mu + \delta)) + a\mu\beta du_1 S_* I_*$.

Stability of disease free equilibrium E_1

Theorem

For all time delays $\tau_1, \tau_2 \geq 0$, the disease free equilibrium E_1 of the system (1) is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$.

Stability of the Infected Equilibrium E_2

The linearized system corresponding to the delay system (1) around infected equilibrium E_2 is given as:

$$\frac{dY(t)}{dt} = J_1 Y(t) + J_2 Y(t-\tau_1) + J_3 Y(t-\tau_2). \tag{2}$$

Here $J_1 = \begin{pmatrix} -(\mu + \beta I_* + du_1 Z_*) & -\beta S_* & 0 & -du_1 S_* \\ \beta I_* & 0 & 0 & 0 \\ du_1 Z_* & \gamma & -\mu & du_1 S_* \\ 0 & 0 & 0 & -a_0 \end{pmatrix}$,

$J_2 = \begin{pmatrix} 0 & 0 & \delta_0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}$

and $Y(t) = (S(t), I(t), R(t), Z(t))^T$.

Stability of the Infected Equilibrium E_2

The characteristic equation corresponding to the linearized system (2) is given by,

$$D(\lambda, \tau_1, \tau_2) := det(\lambda I_4 - (J_1 + e^{-\lambda\tau_1} J_2 + e^{-\lambda\tau_2} J_3)) = 0,$$

here I_4 is the identity matrix of order four. This can be written as

$$D(\lambda, \tau_1, \tau_2) := \mathcal{P}(\lambda) + e^{-\lambda\tau_1}\mathcal{Q}(\lambda) + e^{-\lambda\tau_2}\mathcal{R}(\lambda) = 0, \tag{3}$$

where $\mathcal{P}(\lambda) = \lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4$, $\mathcal{Q}(\lambda) = B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4$, $\mathcal{R}(\lambda) = C_1\lambda + C_2$. Here,

$$\begin{aligned}
A_1 &= a_0 + 2\mu + \beta I_* + du_1 Z_* \\
A_2 &= a_0\mu + (\mu + a_0)(\mu + \beta I_* + du_1 Z_*) + \beta^2 S_* I_* \\
A_3 &= \beta I_*(a_0 + \mu)(\mu + \delta + \gamma) + \mu a_0(\mu + \beta I_* + du_1 Z_*) \\
A_4 &= \beta a_0 I_* \mu(\mu + \delta + \gamma) \\
B_1 &= \delta_0, B_2 = \delta_0(a_0 + \mu + \beta I_*) \\
B_3 &= a_0\delta_0(\mu + \beta I_*) + \beta I_* \delta_0(\mu + \delta), B_4 = \beta a_0 I_* \delta_0(\mu + \delta) \\
C_1 &= adu_1\beta S_* I_*, C_2 = \mu adu_1\beta S_* I_*.
\end{aligned}$$

In this case the characteristic equation (3) is given by,

$$\begin{aligned} D(\lambda, \tau_1) \quad := \quad & \lambda^4 + A_1\lambda^3 + A_2\lambda^2 + (A_3 + C_1)\lambda + (A_4 + C_2) \\ & + e^{-\lambda\tau_1}(B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4) = 0. \end{aligned} \tag{4}$$

In order to find instability, we put $\lambda = i\omega$ in the above equation. Separating the real and imaginary parts we have

$$\omega^4 - A_2\omega^2 + A_4 + C_2 = (B_2\omega^2 - B_4)\cos\omega\tau_1 + (B_1\omega^3 - B_3\omega)\sin\omega\tau_1.$$

$$(A_3 + C_1)\omega - A_1\omega^3 = (B_1\omega^3 - B_3\omega)\cos\omega\tau_1 - (B_2\omega^2 - B_4)\sin\omega\tau_1.$$

Now, squaring and adding, we get

$$\omega^8 + A_{11}\omega^6 + A_{12}\omega^4 + A_{13}\omega^2 + A_{14} = 0.$$

Here $A_{11} = A_1^2 - 2A_2 - B_1^2, A_{12} = A_2^2 + 2(A_4 + C_2) - 2A_1(A_3 + C_1) - B_2^2 + 2B_1B_3,$ $A_{13} = (A_3 + C_1)^2 - 2A_2(A_4 + C_2) + 2B_2B_4 - B_3^2$ and $A_{14} = (A_4 + C_2)^2 - B_4^2.$
Writing $m = \omega^2$, we have

$$\psi(m) = m^4 + A_{11}m^3 + A_{12}m^2 + A_{13}m + A_{14} = 0. \tag{5}$$

Case-I: $\tau_1 > 0$ and $\tau_2 = 0$

Theorem

The unique infected equilibrium E_2 of the delay system (1) will be locally asymptotically stable for all $\tau_1 > 0$ provided following conditions hold

$$A_{11} > 0, A_{13} > 0, A_{14} > 0 \ \ and \ \ A_{11}A_{12}A_{13} > A_{13}^2 + A_{11}^2 A_{14}.$$

Lemma (4)

The equation (5) has

(i) *at least one positive root (either one or three) if*

   (a) $A_{11} > 0, A_{12} < 0, A_{13} > 0, A_{14} < 0.$
   (b) $A_{11} < 0, A_{12} < 0, A_{13} > 0, A_{14} < 0.$
   (c) $A_{11} < 0, A_{12} > 0, A_{13} > 0, A_{14} < 0.$
   (d) $A_{11} < 0, A_{12} > 0, A_{13} < 0, A_{14} < 0.$

(ii) *exactly one positive root if*

   (a) $A_{11} < 0, A_{12} < 0, A_{13} < 0, A_{14} < 0.$
   (b) $A_{11} > 0, A_{12} < 0, A_{13} < 0, A_{14} < 0.$
   (c) $A_{11} > 0, A_{12} > 0, A_{13} < 0, A_{14} < 0.$
   (d) $A_{11} > 0, A_{12} > 0, A_{13} > 0, A_{14} < 0.$

Hopf Bifurcation

For the existence of the Hopf bifurcation there must be a threshold value of the delay τ_{10} such that:

(H_1) $\lambda_{1,2}(\tau_{10}) = \pm i\omega_{10}(\omega_{10} > 0)$ and all other eigenvalues are with negative real parts at $\tau = \tau_{10}$.

(H_2) $\left[Re\left(\frac{d\lambda_{1,2}}{d\tau_1}\right)^{-1}\right]\Big|_{\lambda = i\omega_{10}} \neq 0.$

Let $m_{10} = \omega_{10}^2$ be a positive root of the equation (5) then $\pm i\omega_{10}$ is a pair of purely imaginary root of the equation (3) for the threshold value of the delay τ_1. The critical value of τ_1 is given as:

$$\tau_{10} = \frac{1}{\omega_{10}}[\arccos(\Upsilon(\omega_{10}))], \tag{6}$$

where

$$\Upsilon(\omega_{10}) = \frac{(B_2\omega_{10}^2 - B_4)(\omega_{10}^4 - A_2\omega_{10}^2 + A_4 + C_2) + (B_1\omega_{10}^3 - B_3\omega_{10})((A_3 + C_1)\omega_{10}}{(B_2\omega_{10}^2 - B_4)^2 + (B_1\omega_{10}^3 - B_3\omega_{10})^2} \tag{7}$$

Hopf Bifurcation

The transversality condition (H_2) gives

$$\left[Re \left(\frac{d\lambda}{d\tau_1} \right)^{-1} \right] \Bigg|_{\lambda = i\omega_{10}} = \frac{\psi^{'}(m)}{(B_2\omega_{10}^2 - B_4)^2 + (B_1\omega_{10}^3 - B_3\omega_{10})^2}. \qquad (8)$$

Lemma

Let $i\omega_{10}$ be a purely imaginary root with $m_{10} = \omega_{10}^2$ such that $\psi(\omega_{10}) = 0$ and $\psi^{'}(\omega_{10}) \neq 0$, then $\left[Re \left(\frac{d\lambda}{d\tau_1} \right)^{-1} \right] \Big|_{\lambda = i\omega_{10}} \neq 0$ and its sign is the same as $\psi^{'}(\omega_{10}) \neq 0$.

Thus transversality condition holds.

Theorem (6)

The unique infected equilibrium E_2 is locally asymptotically stable for $\tau_1 < \tau_{10}$ and is unstable for $\tau_1 > \tau_{10}$. At $\tau_1 = \tau_{10}$, a Hopf bifurcation occurs, i.e. a family of periodic solutions bifurcates from the infected equilibrium E_2 as delay parameter τ_1 crosses the threshold value τ_{10} [1, 2].

Numerical Results

We consider a set of representative parameters as: $\Lambda = 10, \beta = 0.0325, \mu = 0.04, d = 0.17, \delta = 0.5, \delta_0 = 0.5, a = 0.1, a_0 = 0.1, \gamma = 0.1, u_1 = 0.9$.

The model system has the unique infected equilibrium $E_2 = (19.69, 11.95, 68.91, 11.95)$ along with disease free equilibrium $E_1 = (250, 0, 0, 0)$ whereas $R_0 = 12.69 > 1$.

A_{11}, A_{12}, A_{13} and A_{14} satisfy the condition $i(a)$ of Lemma 4. Thus the equation (5) has one positive root (0.0546). The characteristic equation (4) has a pair of purely imaginary root $\pm 0.233i$ with $\omega_{10} = 0.233$, $\tau_{10} = 8.73$ and the transversality condition $\left[Re \left(\frac{d\lambda}{d\tau_1} \right)^{-1} \right] \Big|_{\lambda = i\omega_{10}} = 2.308 > 0$ also holds. Thus from Theorem 6 the delay system (1) will be stable for $\tau_1 \in [0, \tau_{10})$ and unstable for $\tau_1 > \tau_{10}$.

Periodic oscillations bifurcate near unique infected equilibrium E_2 as τ_1 crosses τ_{10}.

Solution Trajectories for $\tau_1 = 8.45$ and $\tau_1 = 9.6$

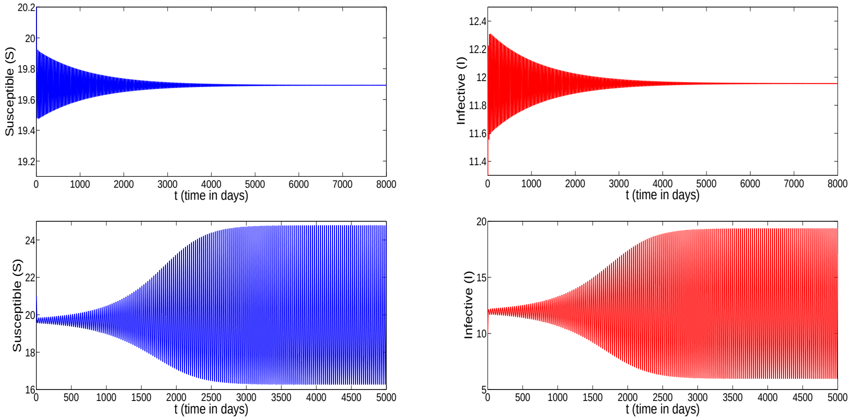


Figure : (a) Solution trajectory of susceptible population showing stability for $\tau_1 = 8.45$.
(b) Solution trajectory of infective population showing stability for $\tau_1 = 8.45$.
(c) Oscillation in susceptible population for $\tau_1 = 9.6$.
(d) Oscillation in infective population for $\tau_1 = 9.6$.

Bifurcation diagrams

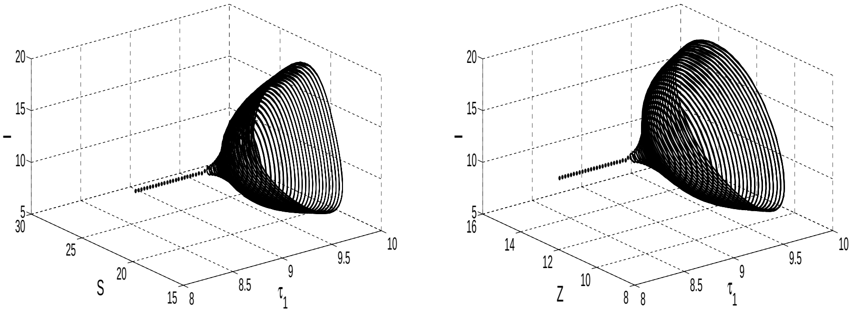


Figure : Plot for the bifurcation diagram showing the occurrence of periodic orbits when τ_1 crosses $\tau_{10} = 8.45$ in I-R and Z-R planes.

Similarly as above we analyse the characteristic equation

$$D(\lambda, \tau_2) := \lambda^4 + P_1\lambda^3 + Q_1\lambda^2 + W_1\lambda + D_1 + e^{-\lambda\tau_2}(C_1\lambda + C_2) = 0. \qquad (9)$$

We ge the equation in $m = \omega^2$ as

$$\Psi(m) = m^4 + B_{11}m^3 + B_{12}m^2 + B_{13}m + B_{14} = 0. \qquad (10)$$

Here $B_{11} = P_1^2 - 2Q_1, B_{12} = Q_1^2 + 2D_1 - 2P_1W_1, B_{13} = W_1^2 - 2D_1Q_1 - C_1^2$ and $B_{14} = D_1^2 - C_2^2$, and $P_1 = A_1 + B_1$, $Q_1 = A_2 + B_2$, $W_1 = A_3 + B_3$ and $D_1 = A_4 + B_4$.

Theorem

The unique infected equilibrium E_2 of the delay system (1) will be locally asymptotically stable for all $\tau_2 > 0$ provided following conditions hold

$$B_{11} > 0, B_{13} > 0, B_{14} > 0 \ \ \textit{and} \ \ B_{11}B_{12}B_{13} > B_{13}^2 + B_{11}^2B_{14}.$$

Case-II: $\tau_1 = 0$ and $\tau_2 > 0$

Let $m_{20} = \omega_{20}^2$ be a positive root of the equation (10) then $\pm i\omega_{20}$ is a pair of purely imaginary root of the equation $D(\lambda, \tau_2) = 0$ for the threshold value of the delay τ_{20}:

$$\tau_{20} = \frac{1}{\omega_{20}}[\arccos(\Phi(\omega_{20}))], \tag{11}$$

where,

$$\Phi(\omega_{20}) = \frac{\omega_{20}^4(P_1C_1 - C_2) + \omega_{20}^2(C_2Q_1 - C_1W_1) - C_2D_1}{C_2^2 + C_1^2\omega_{20}^2}. \tag{12}$$

The transversality condition (H_2) is

$$\left[Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\right]\Bigg|_{\lambda=i\omega_{20}} = \frac{\Psi^{'}(m)}{C_2^2\omega_{20}^2 + C_1^2}. \tag{13}$$

Lemma

Let $i\omega_{20}$ be a purely imaginary root with $m_{20} = \omega_{20}^2$ such that $\psi(\omega_{20}) = 0$ and $\Psi^{'}(\omega_{20}) \neq 0$, then $\left[Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\right]\Big|_{\lambda=i\omega_{20}} \neq 0$ and its sign is the same as $\Psi^{'}(\omega_{20}) \neq 0$.

Theorem (9)

The unique infected equilibrium E_2 *is locally asymptotically stable for* $\tau_2 < \tau_{20}$ *and is unstable for* $\tau_2 > \tau_{20}$*. At* $\tau_2 = \tau_{20}$*, a Hopf bifurcation occurs, i.e. a family of periodic solutions bifurcates from the infected equilibrium* E_2 *as delay parameter* τ_2 *crosses the threshold value* τ_{20} *[1, 2].*

Consider $\Lambda = 0.2, \beta = 0.0001, \mu = 0.00004, d = 0.017, \delta = 0.01, \delta_0 = 0.1, a = 0.1, a_0 = 0.06, \gamma = 0.1, u_1 = 0.9$ along with initial population size: $S(0) = 1100, I(0) = 10, R(0) = 2050$ and $Z(0) = 11$.

For this $R_0 = 4.54 > 1$ and hence the delay model system has a unique infected equilibrium $E_2 = (1100.04, 7.32, 2061.44, 12.2)$ along with disease free equilibrium $E_1 = (5000, 0, 0, 0)$.

B_{11}, B_{12} are positive and B_{13}, B_{14} are negative, and satisfy the condition ii(c) of Lemma 4. The equation (10) has exactly one positive root (0.002764). We find that the characteristic equation (9) have a pair of purely imaginary root $\pm 0.0528i$ with $\omega_{20} = 0.0528$ and $\tau_{20} = 12.78$ and the transversality condition $\left[Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\right]\Big|_{\lambda = i\omega_{20}} = 1.47 > 0$ also holds.

From Theorem 9, we conclude that periodic oscillations bifurcate near the unique infected equilibrium E_2 as τ_2 crosses τ_{20}.

Solution trajectories for $\tau_2 = 12.5$ and $\tau_2 = 13.5$

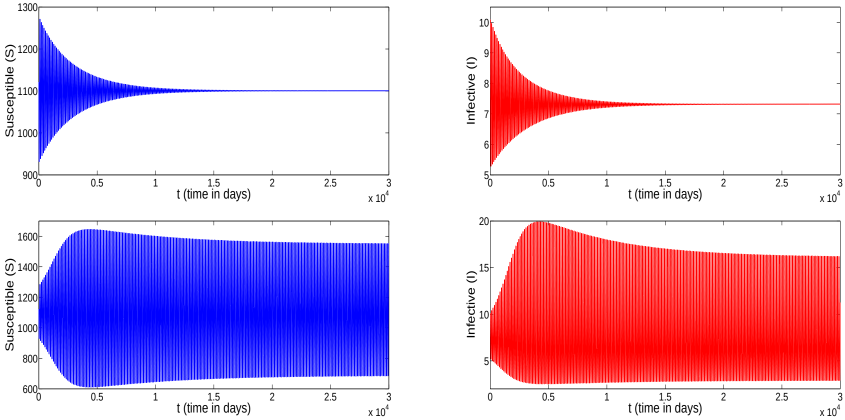
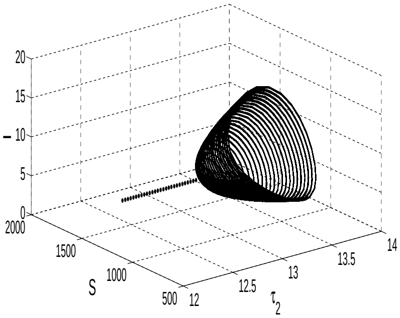


Figure : (a) Solution trajectory of susceptible population showing stability for $\tau_2 = 12.5$.
(b) Solution trajectory of infective population showing stability for $\tau_2 = 12.5$
(c) Oscillation in susceptible population for $\tau_2 = 13.5$.
(d) Oscillation in infective population for $\tau_2 = 13.5$.

Bifurcation diagram



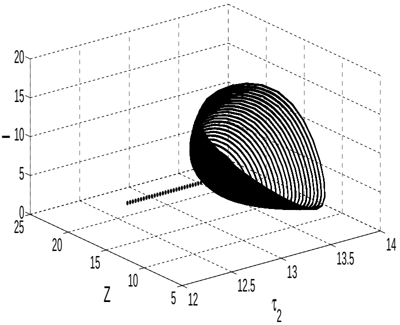


Figure : Plot for the bifurcation diagram showing the occurrence of periodic orbits when τ_2 crosses $\tau_{20} = 12.78$ in S-I and Z-I planes.

Assume that the equation (10) has two positive roots $m_{21} = \omega_{2_1}^2$ and $m_{22} = \omega_{2_2}^2$. The corresponding two pair of purely imaginary roots of the equation $D(\lambda, \tau_2) = 0$ are $\pm i\omega_{21}$ and $\pm i\omega_{22}$. Now, the corresponding threshold values for the delay τ_2 are as follows:

$$\tau_{2_j} = \frac{1}{\omega_{2_j}}[\arccos(\Phi(\omega_{2_j}))], \quad j = 1, 2. \tag{14}$$

Where $\Phi(\omega_{2_j}) = \frac{\omega_{2_j}^4(P_1C_1 - C_2) + \omega_{2_j}^2(C_2Q_1 - C_1W_1) - C_2D_1}{C_2^2 + C_1^2\omega_{2_j}^2}$, j=1, 2. The corresponding transversality conditions are given by,

$$\left[Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\right]\Bigg|_{\lambda = i\omega_{2_j}} = \frac{\Psi^{'}(m)}{C_2^2\omega_{2_j}^2 + C_1^2}, j = 1, 2. \tag{15}$$

Existence of Hopf-Hopf Bifurcation

We state the following result using the results given in [3, 4, 5].

Lemma (10)

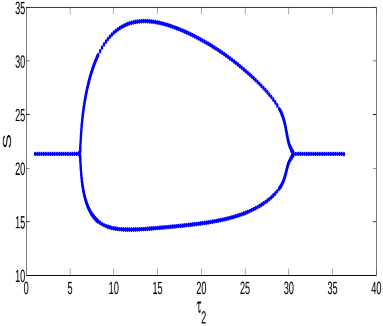
Let $i\omega_{2_j}$ be the purely imaginary roots with $m_{2_j} = \omega_{2_j}^2$ such that $\psi(\omega_{2_j}) = 0$ and $\Psi'(\omega_{2_j}) \neq 0$, j=1, 2, then $\left[Re \left(\frac{d\lambda}{d\tau_2} \right)^{-1} \right] \Big|_{\lambda = i\omega_{21}} > 0$ if $\Psi'(\omega_{21}) > 0$ and $\left[Re \left(\frac{d\lambda}{d\tau_2} \right)^{-1} \right] \Big|_{\lambda = i\omega_{22}} < 0$ if $\Psi'(\omega_{22}) < 0$.

The following result ensures the occurrence of Hopf-Hopf (double) bifurcation following the above Lemma 10.

Theorem (9)

The delay model undergoes Hopf-Hopf (double) bifurcation at τ_{21} and τ_{22} respectively. The unique infected equilibrium E_2 is locally asymptotically stable for $\tau_2 < \tau_{21}$. At $\tau_2 = \tau_{21}$ the system undergoes Hopf bifurcation and the infected equilibrium E_2 loses its stability. Further as delay parameter τ_2 passes, the infected equilibrium E_2 regains its stability at $\tau_2 = \tau_{22}$ and again system undergoes Hopf bifurcation. The infected equilibrium E_2 remains stable for $\tau_2 > \tau_{22}$.

Hopf-Hopf bifurcation diagram



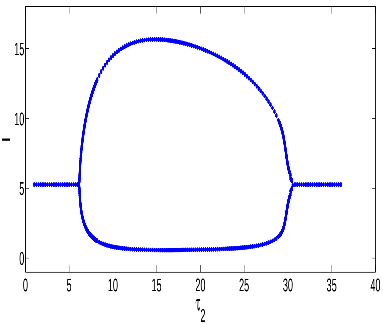


Figure : Plot for the Hopf-Hopf bifurcation diagram for susceptible and infective population respectively.

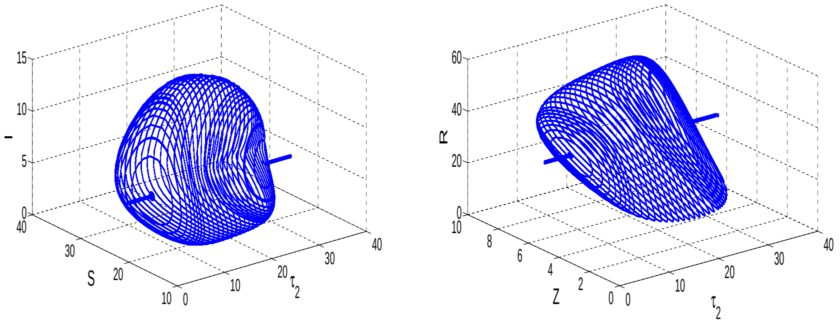


Figure : Plot for the Hopf-Hopf bifurcation diagram in S-I and R-Z planes.

Solution trajectories if I populations as τ_2 vary

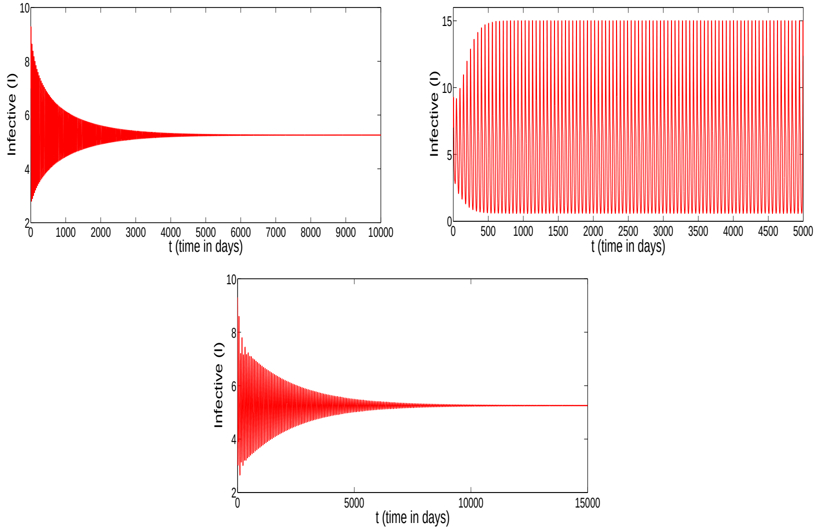
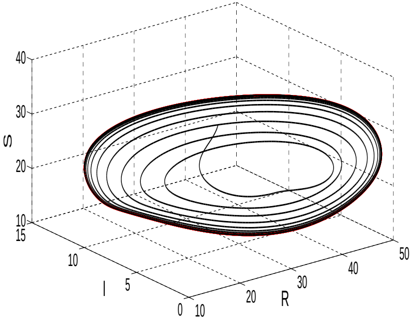


Figure : Profiles of infective population showing (a) stability of E_2 for $\tau_2 = 6 < \tau_{20} = 6.18$. (b) occurrence of oscillation around E_2 for $\tau_2 = 20$. (c) stability of E_2 for $\tau_2 = 31 > \tau_{20} = 29.79$.

Periodic solutions



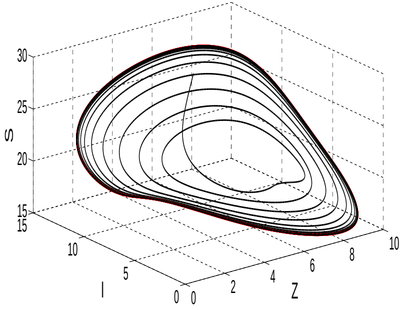


Figure : Trajectories approaching to periodic orbit for $\tau_2 = 15$.

Case-III: $\tau_1, \tau_2 > 0$

Lemma

If all roots of equation $D(\lambda, 0, \tau_2) = 0$ have negative real parts for all $\tau_2 > 0$, then there exists a $\tau_{10}(\tau_2) > 0$ such that all roots of equation (3) have negative real parts when $\tau_1 < \tau_{10}(\tau_2)$.

Theorem

Suppose $R_0 > 1$ and the conditions of Theorem 9 hold. Let $\tau_{20} = \frac{1}{\omega_{20}}[\arccos(\Phi(\omega_{20}))]$, where $\Phi(\omega_{20})$ is as given in equation (12). Then for any $\tau_2 < \tau_{20}$ there exists a $\tau_{10}(\tau_2) > 0$ such that the unique infected equilibrium E_2 is locally asymptotically stable for $\tau_1 < \tau_{10}(\tau_2)$.

Theorem

Suppose $R_0 > 1$ and the conditions of Theorem 6 hold. Let $\tau_{10} = \frac{1}{\omega_{10}}[\arccos(\Upsilon(\omega_{10}))]$, where $\Upsilon(\omega_{10})$ is as given in equation (7). Then for any $\tau_1 < \tau_{10}$ there exists a $\tau_{20}(\tau_1) > 0$ such that the unique infected equilibrium E_2 is locally asymptotically stable for $\tau_2 < \tau_{20}(\tau_1)$.

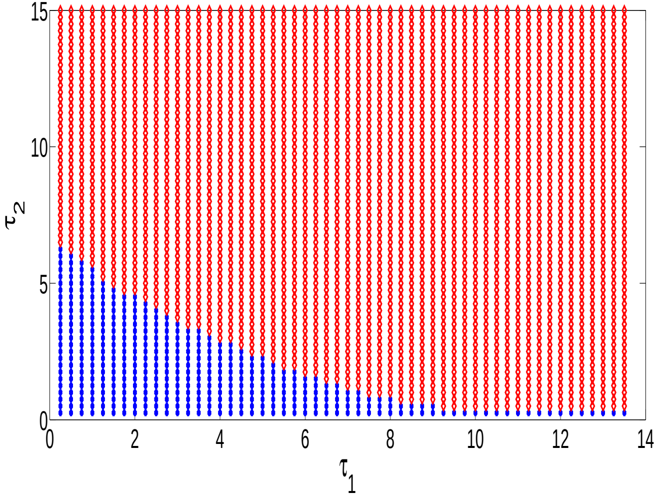


Figure : Plot of stability region for the delay system when both the delays are present. 'asterisk' shows the stable infected equilibrium for the values of τ_1 and τ_2, and 'diamond' otherwise.

- A nonlinear delay differential equation model SIRS for the disease dynamics is proposed and analyzed which accounts for the effect of human behavioral response due to delayed information about the disease prevalence and also the delayed impact of immunity loss related with protection.
- Model analysis is carried out and found that when $R_0 < 1$ the disease free equilibrium is locally stable irrespective of the delay effect.
- Existence of Hopf bifurcation around the unique infected equilibrium is observed as the time delay crosses a threshold value for both delays.
- Hence the delay in reporting of infective as well as delay in waning the immunity destabilises the system and causes the occurrence of oscillations.
- We also observed the occurrence of Hopf-Hopf (double) bifurcation at two different delays for the model system. Hence the delay effect in the model system may cause multiple stability switches.
- Hence the effect of delays shows rich and complex dynamics in the model and provides important insight.

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