«THE INVERSE MAGNETOENCEPHALOGRAPHY PROBLEM AND ITS FLAT APPROXIMATION»

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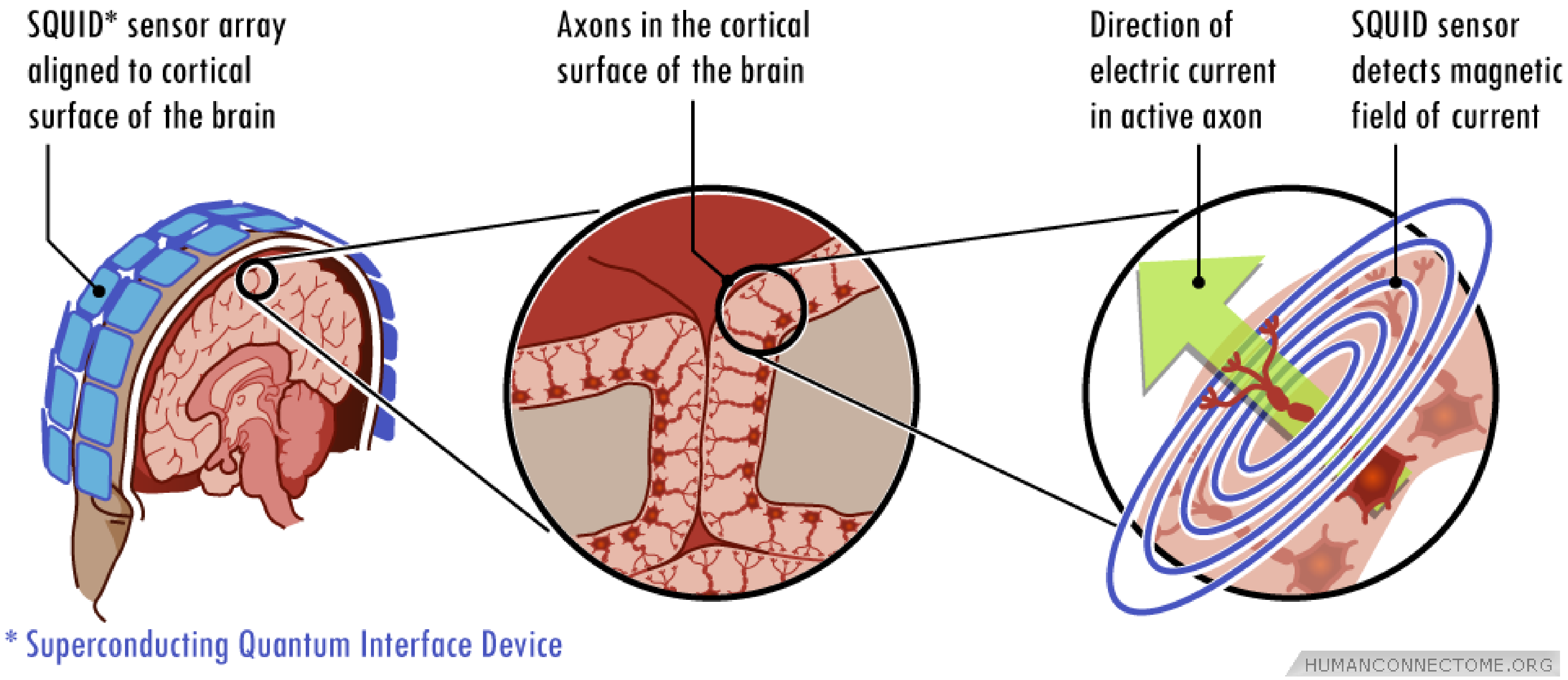
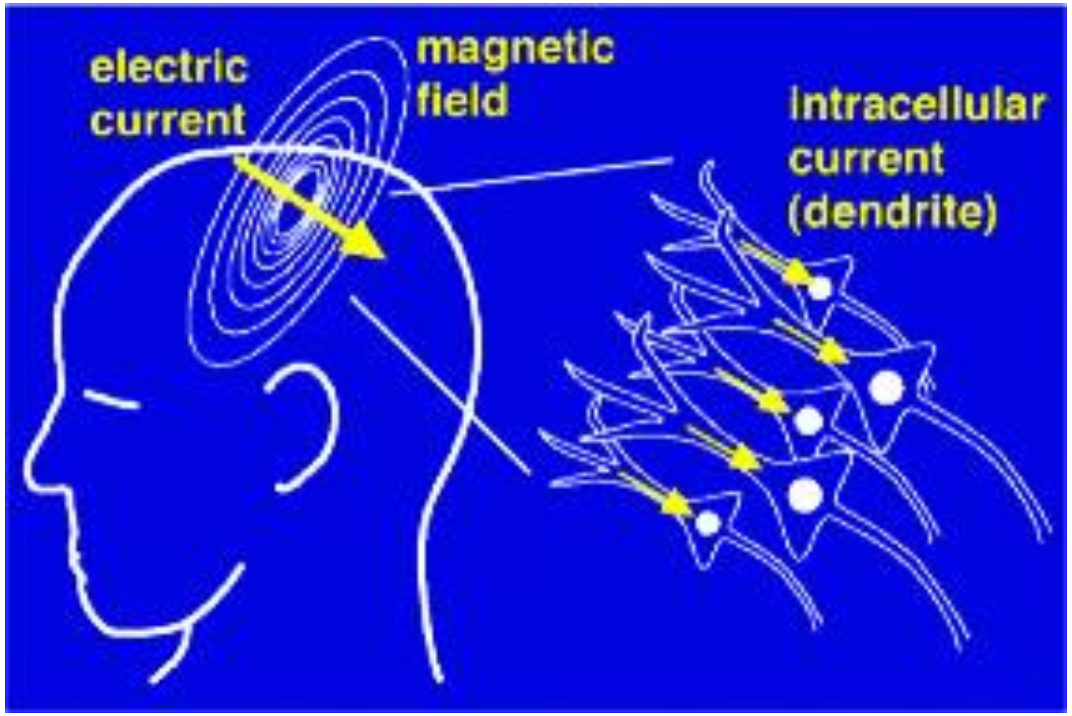
What is MEG?



Magnetoencephalography is a noninvasive technique for investigating neuronal activity in the living human brain.

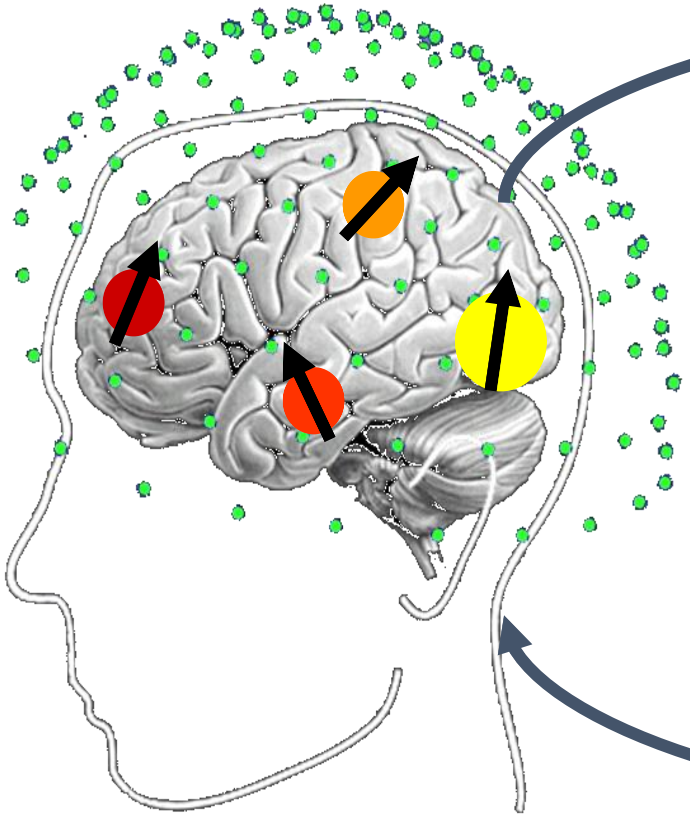
Inverse problem

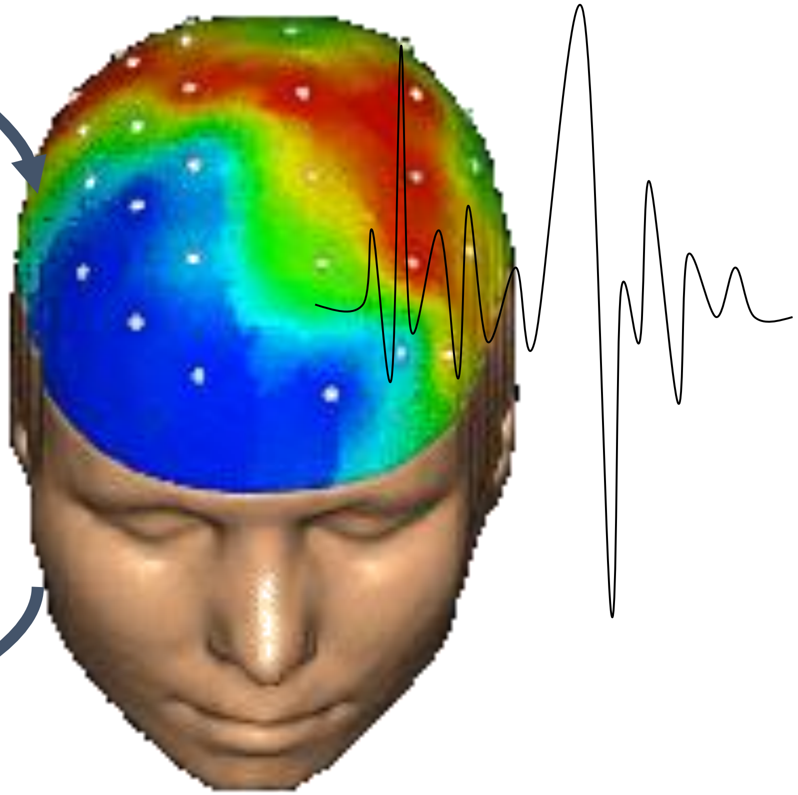
It is a problem of finding distribution of electrical impulses in some area Y, associated with cortex that based on data of its induced magnetic field in another place X that we get by MEG system.



The reason of our interest in this problem

Forward computation





Inverse computation

Start with Maxwell's equations

$$\mathrm{rot}\mathbf{E} = 0,$$
$$\mathrm{rot}\mathbf{H} = \mathbf{J}^{v} + \mathbf{q},$$
$$\mathrm{div}\mathbf{B} = 0,$$
$$\mathrm{div}\boldsymbol{D} = \rho$$

H and **E -** intensity of magnetic and electric fields

$$\mathbf{J}^{v} = \sigma\,\mathbf{E}$$
$$\mathbf{B} = \mu\mathbf{H}$$
$$\boldsymbol{D} = \varepsilon\mathbf{E}$$

$\sigma = \sigma(x) \geq 0$ - conductivity coefficient

$\mu = \mu(x) \geq 0$ - magnetic permeability

$\varepsilon = \varepsilon(x) \geq 0$ - electrical permeability

GENERAL ASPECTS

What we can derive from Maxwell's equations?

$$\mathrm{rot}\mathbf{E} = 0 \quad \Leftrightarrow \quad \mathbf{E} = -\nabla\Phi\,, \qquad \text{and} \qquad \mathrm{div}\mathbf{B} = 0 \quad \Leftrightarrow \quad \mathbf{B} = \mathrm{rot}\mathbf{A}\,.$$

Since $\mathrm{div}(\varepsilon E) = \rho$ $then$ $-\varepsilon\Delta\Phi - \nabla\varepsilon\nabla\Phi = \rho$ *. We get the following formula:*

$$\Delta\mathbf{A}(\mathbf{x}) = -\mathbf{q}(\mathbf{x}) + \nabla\Big[\sigma(\mathbf{x})\Phi(\mathbf{x}) + \mathrm{div}\mathbf{A}(\mathbf{x})\Big] - \Phi(\mathbf{x})\nabla\sigma(\mathbf{x})\,.$$

Considering invariance invariance with respect to a scalar field, we derive this formula: $\Delta\mathbf{A}(\mathbf{x}) = -\mathbf{F}(\mathbf{x})\,, \qquad \text{where} \qquad \mathbf{F}(\mathbf{x}) = \mathbf{q}(\mathbf{x}) + \Phi(\mathbf{x})\nabla\sigma(\mathbf{x})\,.$

Assuming $\mathbf{a}=(\mathrm{a}_1, a_2, a_3), where \Delta a_j(\mathbf{x}) = \delta(\mathbf{x}),\ a_j(\infty) = 0,$ i.e. $a_j(\mathbf{x}) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}|},$ *we get*

$$\Delta\mathbf{A}(\mathbf{x}) = -\int_Y \mathbf{F}(\mathbf{y})\Delta\mathbf{a}(\mathbf{x}-\mathbf{y})\,d\mathbf{y} = \Delta\left[-\int_Y \mathbf{F}(\mathbf{y})\mathbf{a}(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\right].$$

What we can derive from Maxwell's equations?

$$\int\limits_{Y} \frac{\Phi(\mathbf{y})\nabla\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y} = (\sigma_+ - \sigma_0)\mathbf{n}_X \int\limits_{X} \frac{\Phi_0(\mathbf{y}_X)\,d\mathbf{y}_X}{|\mathbf{x}-\mathbf{y}_X|} + (\sigma_0 - \sigma_-)\mathbf{n}_S \int\limits_{S} \frac{\Phi_0(\mathbf{y}_S)\,d\mathbf{y}_S}{|\mathbf{x}-\mathbf{y}_S|}$$

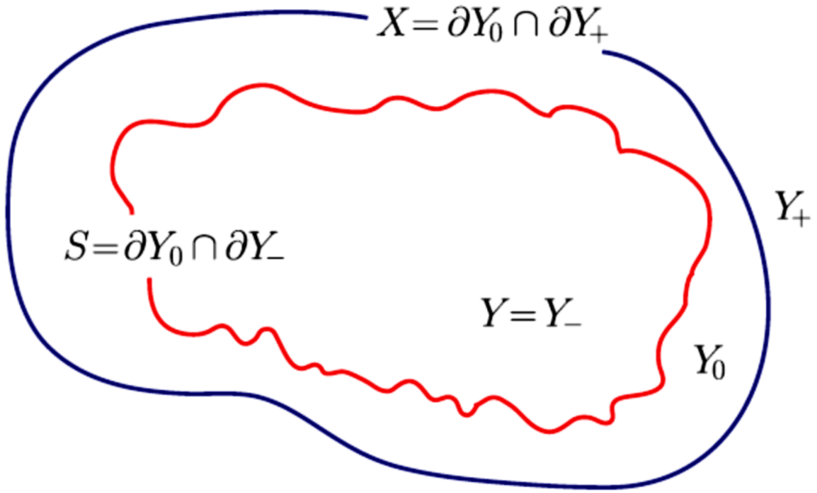
$\mathbf{n}_X$ and $\mathbf{n}_S$ are the outward unit normals to $X = Y_0 \cap Y_+ = Y_+$ and $S = Y_0 \cap Y_- = Y$.

As a result, we obtain an integral equation of the I-kind

$$\mathfrak{I} : \mathbf{q} \mapsto \mathfrak{I}\mathbf{q} \stackrel{def}{=} \int_Y \frac{\mathbf{q}(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in Y,$$

whose right-hand side, given by the formula

$$\mathbf{f}(\mathbf{x}) = 4\pi\mathbf{A}(\mathbf{x}) - \int\limits_{Y} \frac{\Phi(\mathbf{y})\nabla\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y},$$



Theorem 1 (see [3]) *The equation* (8) *uniquely solvable, and its solution has the form*

$$\mathbf{q}(\mathbf{x}) = \mathbf{q}_0(\mathbf{x}) + \mathbf{p}_0(\mathbf{y}')\delta\Big|_{\partial Y},$$

where $\delta\Big|_{\partial Y}$ *is the* δ*-function on* ∂Y, *and* $\mathbf{q}_0 \in C^\infty(\overline{Y})$, $\mathbf{p}_0 \in C^\infty(\partial Y)$ *if* $\mathbf{f} \in C^\infty(\overline{Y})$.[3]

$$\mathfrak{I} : \mathbf{q} \mapsto \mathfrak{I}\mathbf{q} \stackrel{def}{=} \int_Y \frac{\mathbf{q}(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in Y, \tag{8}$$

[3] A.S. Demidov (1973) Elliptic pseudodifferential boundary value problems with a small parameter in
the coefficient of the leading operator, Math. USSR-Sb., 20:3, 439–463.

FLAT APPROXIMATION

According to Biot–Savart law $\mathbf{B}(\mathbf{x}) = \frac{\boldsymbol{Q}\times(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}$

$$B(x) = \int_Y K(x,y)\,Q(y)\,dy \quad , \quad x = (x_1, x_2, x_3) \in X$$

$$K(x,y) = \frac{\mu}{4\pi}\begin{pmatrix} 0 & K_{12}(x,y) & -K_{31}(x,y) \\ -K_{12}(x,y) & 0 & K_{23}(x,y) \\ K_{31}(x,y) & -K_{23}(x,y) & 0 \end{pmatrix}$$

$$K_{12}(x,y) = \frac{x_3 - y_3}{|x-y|^3}\,, \qquad K_{31}(x,y) = \frac{x_2 - y_2}{|x-y|^3}\,, \qquad K_{23}(x,y) = \frac{x_1 - y_1}{|x-y|^3}$$

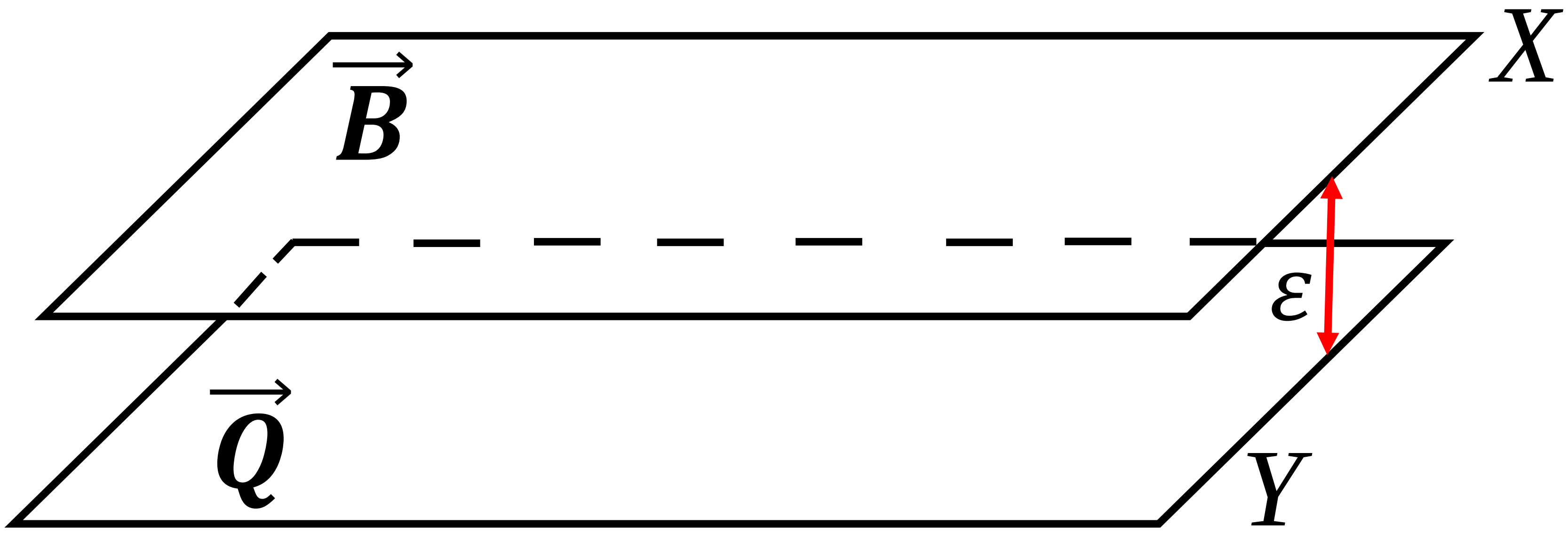
In first time we observe a following flat model:

$$X = \mathbb{R}_2 \ni x = (x_1, x_2), |x_k| < \infty$$

$$(\mathbb{R}_3 \supset Y) \ni y = (y_1, y_2, -\varepsilon): |y_k| < \infty$$

ε = 1

The equation assumes the following form:



$$\sum_{m=1}^{3} \int_{Y} K_{lm}(x-y) Q_m(y) dy = B_l(x), \qquad l = 1,2,3$$

We rewrite our equation in the following form:

$$(1)\ Op\left(\widetilde{K}(\xi)\right) Q(z) = B(x),$$ **where**

$$Op\left(\widetilde{K}(\xi)\right) = \mathcal{F}^{-1}_{\xi\to x}\widetilde{K}(\xi)\mathcal{F}_{z\to\xi}\ ,$$

$$\widetilde{K}(\xi)=\mathcal{F}_{s\to\xi}\ K(s)$$

$\widetilde{K}(\xi)$- **is a symbol of a pseudodifferential operator**

$$(2)\ \widetilde{K}(\xi)\tilde{Q}(\xi) = \tilde{B}(\xi), \xi = (\xi_1, \xi_2)$$

Lemma 1

$$\widetilde{K}_{12}(\xi) = E(\xi), E(\xi) = 2\pi e^{-2\pi|\xi|} :$$

$\xi = (\xi_1, \xi_2), |\xi| = \sqrt{\xi_1^2 + \xi_2^2}$, **and**

$$\widetilde{K}_{23}(\xi) = -i\frac{\xi_1}{|\xi|}E(\xi), \widetilde{K}_{31}(\xi) = -i\frac{\xi_2}{|\xi|}E(\xi),$$

where $\widetilde{K}(\xi)=\mathcal{F}_{s\to\xi}\, K(s)$

Lemma 2

From the matrix equation $(\mathbf{2})$ $\widetilde{\boldsymbol{K}}(\boldsymbol{\xi})\widetilde{\boldsymbol{Q}}(\boldsymbol{\xi}) = \widetilde{\boldsymbol{B}}(\boldsymbol{\xi})$ **we obtain the following relations:**

$$\tilde{B}_j(\xi) = \mathcal{F}_{x\to\xi}\, B_j(x);\ \tilde{Q}_j(\xi) = \mathcal{F}_{y\to\xi}\, Q_j(y)$$

$$i\xi_2\tilde{B}_2(\xi) + i\xi_1\tilde{B}_1(\xi) = |\xi|\tilde{B}_3(\xi)$$

$$\tilde{Q}_1(\xi) = -\frac{\tilde{B}_2(\xi)}{E(\xi)} - i\,\frac{\xi_1}{|\xi|}\,\tilde{Q}_3(\xi),$$

$$\tilde{Q}_2(\xi) = \frac{\tilde{B}_1(\xi)}{E(\xi)} - i\,\frac{\xi_2}{|\xi|}\,\tilde{Q}_3(\xi)$$

Theorem 1

Let $\tilde{B}_1(\xi)$ **and** $\tilde{B}_2(\xi)$ **be continuous and have compact support.**

Then the vector $Q^B = (A_1(y), A_2(y), 0)$,

where $A_1(y) = \mathcal{F}^{-1}_{\xi\to y}\left(-\frac{\tilde{B}_2(\xi)}{E(\xi)}\right), A_2(y) = \mathcal{F}^{-1}_{\xi\to y}\left(\frac{\tilde{B}_1(\xi)}{E(\xi)}\right)$,

$E(\xi) = 2\pi e^{-2\pi|\xi|}$, **satisfies the equation**

$$B(x) = \int_Y K(x-y)Q(y)dy,$$

the general solution of which is representable in the form $Q = Q^B + Q^0$,

where $Q^0 = (Q_1^0, Q_2^0, Q_3^0); Q_1^0 = -Op\left(i\frac{\xi_1}{|\xi|}\right)Q_3^0(y)$,

$$Q_2^0 = -Op\left(i\frac{\xi_2}{|\xi|}\right)Q_3^0(y), Q_3^0(y) \in L_2.$$

Proposal 1

Suppose that $y_1 = r\cos 2\pi\Theta$, $y_2 = r\sin 2\pi\Theta$.

$$G(r,\Theta) = g(y_1,y_2) = \sum_{m\in\mathbb{Z}} G_m(r)e^{i2\pi m\Theta},$$

where $G_m(r) \in \mathbb{C}$. **Then**

$$\tilde{G}(|\xi|,\omega) = \sum_{n\in\mathbb{Z}} e^{i2\pi\left(\omega-\frac{1}{4}\right)n} \int_0^\infty rG_n(r)J_n(2\pi|\xi|r)dr\,,$$

where $\tilde{G}(|\xi|,\omega) = \mathcal{F}_{\,y\to\xi}\, g(y)$,

$$\xi_1 = |\xi|\cos 2\pi\omega\,, \xi_2 = |\xi|\sin 2\pi\omega$$

Proposal 2

Let $\xi_1 = |\xi| \cos 2\pi\omega, \xi_2 = |\xi| \sin 2\pi\omega$.

$$\tilde{C}(|\xi|, \omega) = \tilde{c}(\xi_1, \xi_2) = \sum_{m \in \mathbb{Z}} \tilde{C}_m(|\xi|) e^{-i2\pi m\omega},$$

where $\tilde{C}_m(|\xi|) \in \mathbb{C}$. **Then**

$$\mathcal{F}^{-1}_{\xi \to y} \tilde{c}(\xi_1, \xi_2) = \sum_{n \in \mathbb{Z}} e^{-i2\pi\left(\Theta - \frac{1}{4}\right)n} \int_0^\infty |\xi| \tilde{C}_n(|\xi|) J_n(2\pi|\xi|r) d|\xi|$$

where $\mathrm{y} = (y_1, y_2)$: $y_1 = r \cos 2\pi\Theta, y_2 = r \sin 2\pi\Theta$ и $\xi = (\xi_1, \xi_2)$.

Final result

Let $x = (\rho\cos 2\pi\varphi, \rho\sin 2\pi\varphi)$, **and** $\widetilde{\mathrm{B}}_{\mathrm{k}}(\xi)|_{\mathrm{k}=1,2} = \sum_{\mathrm{n}\in\mathbb{Z}} \tilde{\mathrm{C}}_{\mathrm{n}}(|\xi|)\mathrm{e}^{-\mathrm{i}2\pi\mathrm{n}\omega}$,

where $\tilde{\mathrm{C}}_n(|\xi|) = p_n^k(|\xi|) + q_n^k(|\xi|)$ **are such that** $\tilde{B}_k(\xi) = 0$ **for** $|\xi| > R$ **and some** $R > 0$

and let the following condition be satisfied:

$$\sum_{l\in\mathbb{Z}}(-1)^l\int_0^\infty |\xi|\{J_{2l}(q_{2l}^k\cos 4\pi l\varphi - p_{2l}^k\sin 4\pi l\varphi) + J_{2l+1}(p_{2l+1}^k\cos 2\pi(2l+1)\varphi - q_{2l}^k\sin 2\pi(2l+1)\varphi)\}d|\xi| = 0$$

Final result

Then the required components of the magnetic field, which are $\mathcal{F}^{-1}_{\xi\to x}\tilde{B}_k(\xi)$, are given by the following explicit formulas:

$$B_{k=1,2}(x) = \sum_{l\in\mathbb{Z}}(-1)^l\int_0^\infty |\xi|\{J_{2l}\left(p^k_{2l}\cos 4\pi l\varphi + q^k_{2l}\sin 4\pi l\varphi\right) + J_{2l+1}\left(-q^k_{2l+1}\cos 2\pi(2l+1)\varphi + p^k_{2l}\sin 2\pi(2l+1)\varphi\right)\}d|\xi|$$

a $B_3(x) = \frac{1}{4\pi^2}\int\frac{\partial_{y_1}B_1(y)+\partial_{y_2}B_2(y)}{|x-y|}dy.$

Calculus

The required vector of the magnetic field must deliver the minimum to the following functional:

$$\Phi(\tilde{\mathbf{B}}(\mathbf{x})) = \sum_{j=1}^{3} \sum_{k=(k_1,k_2)} \left| \mathbf{F}^{-1}_{\xi \to x_k} \tilde{\mathbf{B}}(x_k) - \mathbf{B}(x_k) \right|$$

We consider the special case when $p_n^k(|\xi|) = 0, q_n^k(|\xi|) = 0 : n \neq 0, k = 1, 2$

So we get the following expression $B_k(\mathbf{x}) = \int\limits_0^\infty |\xi| J_0(2\pi|\xi|\rho) p_0^k(|\xi|) d|\xi|$

For simplicity of calculations, we consider the following function $\mathrm{p}_0^k(|\xi|) = \frac{e^{-2\pi|\xi|}}{|\xi|}$

In this case, $B_k(x)$ takes the following form: $B_k(\mathbf{x}) = \frac{1}{2\pi\sqrt{1+\rho^2}}, (k = 1, 2).$

[1] M. Yu. Shatalov, AS Demidov, IA Fedotov, THEORETICAL FOUNDATIONS OF CHEMICAL TECHNOLOGY, Vol. 50, No. 2, p. 1-11 (2016).

Calculus

Taking into account $\mathbf{div}\mathbf{B} = \mathbf{0}$, $B_3 = 0$.

Using the technique presented in [1], we obtain the following: $p_0^k(|\xi|, \lambda) = \dfrac{\lambda e^{-2\pi|\xi|}}{|\xi|}$

$$B_k(\mathbf{x}, \lambda) = \frac{\lambda}{\mathbf{2}\pi\sqrt{\mathbf{1} + \rho^{\mathbf{2}}}} = \lambda \mathbf{B_k}(\mathbf{x}), (\mathbf{k} = \mathbf{1}, \mathbf{2}) \text{ , } B_3 = 0$$

It is easy to see that: $\dfrac{\partial B_l(\mathbf{x_0}, \lambda)}{\partial \lambda} = \lambda B_l(\mathbf{x_0}), (\mathbf{l} = \mathbf{1}, \mathbf{2}, \mathbf{3})$

$$\Phi(\mathbf{x_0}, \lambda) = (\mathbf{B_1}(\mathbf{x_0}, \lambda) - \mathbf{B_1}(\mathbf{x_0}))^{\mathbf{2}} + (\mathbf{B_2}(\mathbf{x_0}, \lambda) - \mathbf{B_2}(\mathbf{x_0}))^{\mathbf{2}} + (\mathbf{B_3}(\mathbf{x_0}, \lambda) - \mathbf{B_3}(\mathbf{x_0}))^{\mathbf{2}}$$

$$\frac{\partial \Phi(\mathbf{x_0}, \lambda)}{\partial \lambda} = 0$$

$$\frac{\partial \Phi(\mathbf{x_0}, \lambda)}{\partial \lambda} = (B_1(\mathbf{x_0}, \lambda) - \mathbf{B_1}(\mathbf{x_0}))\frac{\partial \mathbf{B_1}(\mathbf{x_0}, \lambda)}{\partial \lambda} + (\mathbf{B_2}(\mathbf{x_0}, \lambda) - \mathbf{B_2}(\mathbf{x_0}))\frac{\partial \mathbf{B_2}(\mathbf{x_0}, \lambda)}{\partial \lambda}$$

Calculus

Substituting all the tabs in final expression, taking into account all expressions ,we obtain:

$$2(\lambda - 1)B_1^2(\mathbf{x_0}) = \mathbf{0}$$

It means only that $\lambda = 1$

Thank you for your attention!