

A stable scheme for simulation of incompressible flows in time-dependent domains and hemodynamic applications

Yuri Vassilevski^{1,2,3}

Maxim Olshanskii⁴ **Alexander Danilov**^{1,2,3}
Alexander Lozovskiy¹ **Victoria Salamatova**^{1,2,3}

¹Institute of Numerical Mathematics RAS

²Moscow Institute of Physics and Technology

³Sechenov University

⁴University of Houston

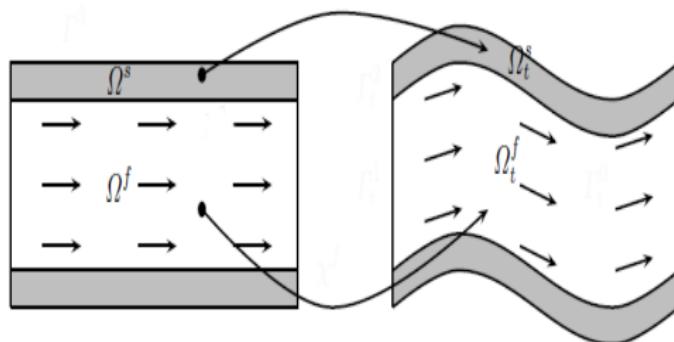
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Fluid-Structure Interaction

Fluid-Structure Interaction problem

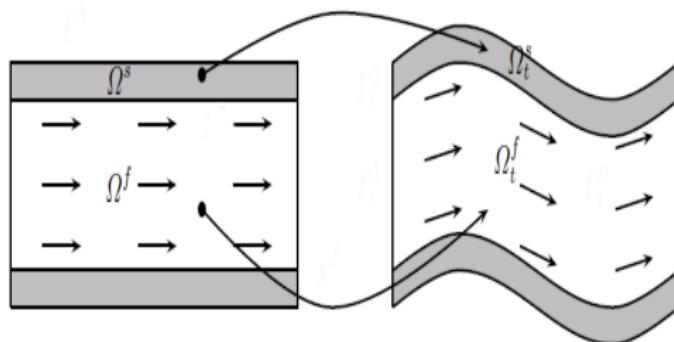
Prerequisites for FSI



- ▶ reference subdomains Ω_f , Ω_s

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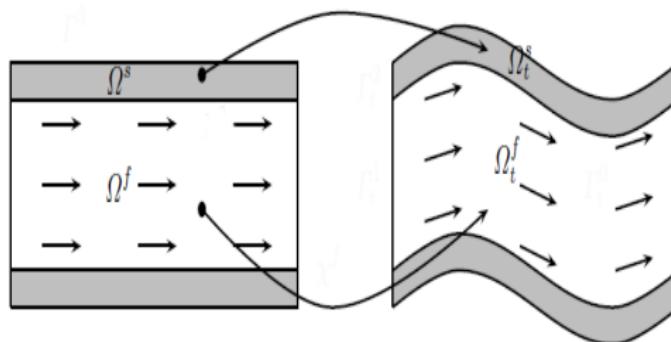
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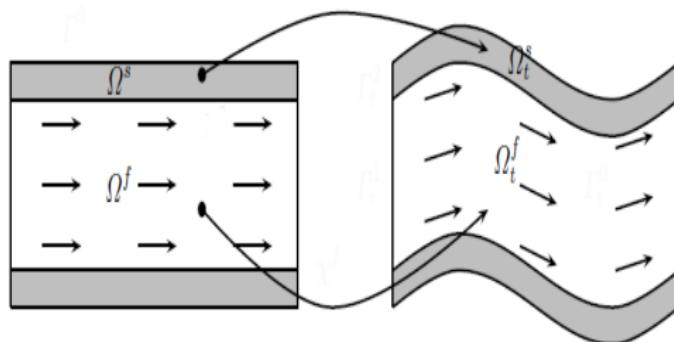
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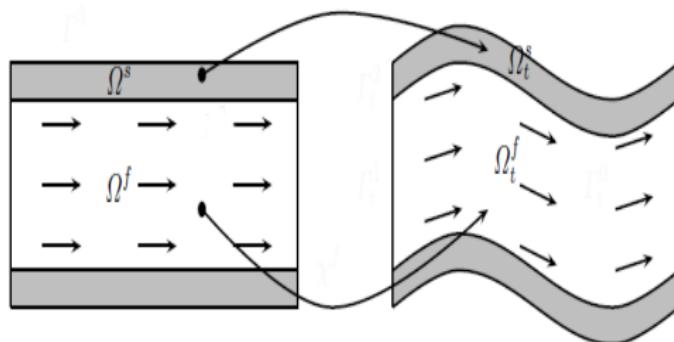
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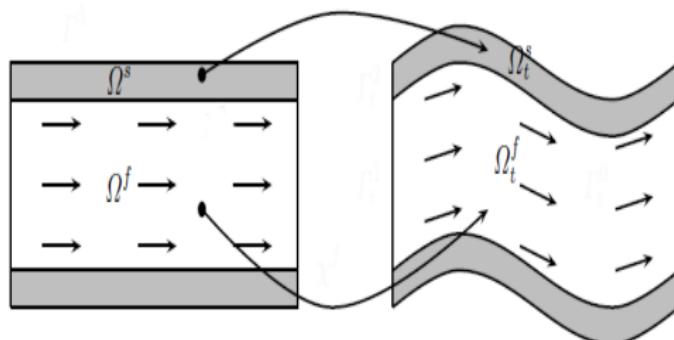
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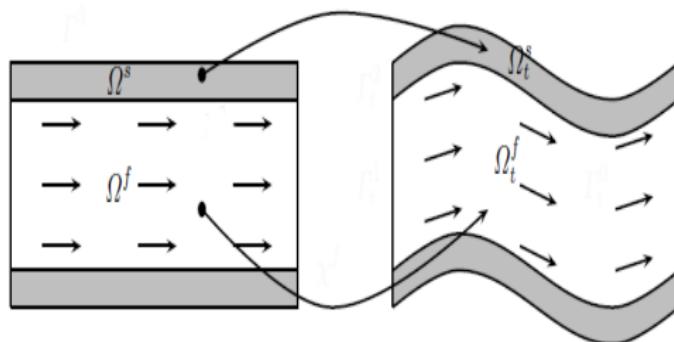
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- ▶ Cauchy stress tensors $\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_s$
- ▶ pressures p_f, p_s
- ▶ density ρ_f is constant

Fluid-Structure Interaction problem

Universal equations in reference subdomains

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{cases} \rho_s^{-1} \operatorname{div}(J\boldsymbol{\sigma}_s \mathbf{F}^{-T}) & \text{in } \Omega_s, \\ (J\rho_f)^{-1} \operatorname{div}(J\boldsymbol{\sigma}_f \mathbf{F}^{-T}) - \nabla \mathbf{v} \left(\mathbf{F}^{-1} \left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} \right) \right) & \text{in } \Omega_f \end{cases}$$

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Kinematic equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v} \quad \text{in } \Omega_s$$

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$$\operatorname{div}(J\mathbf{F}^{-1}\mathbf{v}) = 0 \quad \text{in } \Omega_f \quad \text{or} \quad J\nabla \mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_f$$

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Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f \mathbf{I} + \mu_f ((\nabla \mathbf{v}) \mathbf{F}^{-1} + \mathbf{F}^{-T} (\nabla \mathbf{v})^T) \quad \text{in } \Omega_f$$

FSI problem

User-dependent equations in reference subdomains

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathbf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

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Monolithic approach: Extension of the displacement field to the fluid domain

$$\begin{aligned} G(\mathbf{u}) &= 0 \quad \text{in } \Omega_f, \\ \mathbf{u} &= \mathbf{u}^* \quad \text{on } \partial\Omega_f \end{aligned}$$

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+ Initial, boundary, interface conditions ($\boldsymbol{\sigma}_f \mathbf{F}^{-T} \mathbf{n} = \boldsymbol{\sigma}_s \mathbf{F}^{-T} \mathbf{n}$)

Numerical scheme

- ▶ Conformal triangular or tetrahedral mesh Ω_h in $\widehat{\Omega}$
- ▶ LBB-stable pair for velocity and pressure P_2/P_1
- ▶ Fortran open source software Ani2D, Ani3D (Advanced numerical instruments 2D/3D)

<http://sf.net/p/ani2d/> <http://sf.net/p/ani3d/>:

- ▶ mesh generation
- ▶ FEM systems
- ▶ algebraic solvers

Numerical scheme

Find $\{\mathbf{u}^{k+1}, \mathbf{v}^{k+1}, p^{k+1}\} \in \mathbb{V}_h^0 \times \mathbb{V}_h \times \mathbb{Q}_h$ s.t.

$$\mathbf{v}^{k+1} = \mathbf{g}_h(\cdot, (k+1)\Delta t) \text{ on } \Gamma_{f0}, \quad \left[\frac{\partial \mathbf{u}}{\partial t} \right]_{k+1} = \mathbf{v}^{k+1} \text{ on } \Gamma_{fs}$$

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where

$$\mathbb{V}_h \subset H^1(\widehat{\Omega})^3, \mathbb{Q}_h \subset L^2(\widehat{\Omega}), \mathbb{V}_h^0 = \{\mathbf{v} \in \mathbb{V}_h : \mathbf{v}|_{\Gamma_{s0} \cup \Gamma_{f0}} = \mathbf{0}\}, \mathbb{V}_h^{00} = \{\mathbf{v} \in \mathbb{V}_h^0 : \mathbf{v}|_{\Gamma_{fs}} = \mathbf{0}\}$$

$$\left[\frac{\partial \mathbf{f}}{\partial t} \right]_{k+1} := \frac{3\mathbf{f}^{k+1} - 4\mathbf{f}^k + \mathbf{f}^{k-1}}{2\Delta t}$$

Numerical scheme

$$\begin{aligned} & \int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t} \right]_{k+1} \psi \, d\Omega + \int_{\Omega_s} J_k \mathbf{F}(\tilde{\mathbf{u}}^k) \mathbf{S}(\mathbf{u}^{k+1}, \tilde{\mathbf{u}}^k) : \nabla \psi \, d\Omega + \\ & \int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t} \right]_{k+1} \psi \, d\Omega + \int_{\Omega_f} \rho_f J_k \nabla \mathbf{v}^{k+1} \mathbf{F}^{-1}(\tilde{\mathbf{u}}^k) \left(\tilde{\mathbf{v}}^k - \left[\frac{\partial \mathbf{u}}{\partial t} \right]_k \right) \psi \, d\Omega + \\ & \int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\tilde{\mathbf{u}}^k} \mathbf{v}^{k+1} : \mathbf{D}_{\tilde{\mathbf{u}}^k} \psi \, d\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\tilde{\mathbf{u}}^k) : \nabla \psi \, d\Omega = 0 \quad \forall \psi \in \mathbb{V}_h^0 \end{aligned}$$

$$J_k := J(\tilde{\mathbf{u}}^k), \quad \tilde{\mathbf{f}}^k := 2\mathbf{f}^k - \mathbf{f}^{k-1}, \quad \mathbf{D}_{\mathbf{u}} \mathbf{v} := \{\nabla \mathbf{v} \mathbf{F}^{-1}(\mathbf{u})\}_s, \quad \{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

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$$\int_{\Omega_s} \left[\frac{\partial \mathbf{u}}{\partial t} \right]_{k+1} \phi \, d\Omega - \int_{\Omega_s} \mathbf{v}^{k+1} \phi \, d\Omega + \int_{\Omega_f} G(\mathbf{u}^{k+1}) \phi \, d\Omega = 0 \quad \forall \phi \in \mathbb{V}_h^{00}$$

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Numerical scheme

$$\dots + \int_{\Omega_s} J_k \mathbf{F}(\tilde{\mathbf{u}}^k) \mathbf{S}(\mathbf{u}^{k+1}, \tilde{\mathbf{u}}^k) : \nabla \psi \, d\Omega + \dots$$

- ▶ St. Venant–Kirchhoff model (geometrically nonlinear):

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \lambda_s \text{tr}(\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2)) \mathbf{I} + 2\mu_s \mathbf{E}(\mathbf{u}_1, \mathbf{u}_2);$$

$$\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2) = \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2) - \mathbf{I}\}_s$$

- ▶ inc. Blatz–Ko model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s (\text{tr}(\{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s) \mathbf{I} - \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s)$$

- ▶ inc. Neo-Hookean model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s \mathbf{I}; \quad \mathbf{F}(\tilde{\mathbf{u}}^k) \rightarrow \mathbf{F}(\mathbf{u}^{k+1})$$

$$\{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

The scheme

- ▶ provides strong coupling on interface
- ▶ semi-implicit
- ▶ produces one linear system per time step
- ▶ second order in time

Numerical scheme

The scheme

- ▶ provides strong coupling on interface
- ▶ semi-implicit
- ▶ produces one linear system per time step
- ▶ second order in time
- ▶ unconditionally stable (no CFL restriction), proved with assumptions:
 - ▶ 1st order in time
 - ▶ St. Venant–Kirchhoff inc./comp. (experiment: Neo-Hookean inc./comp.)
 - ▶ extension of \mathbf{u} to Ω_f guarantees $J_k > 0$
 - ▶ Δt is not large

A.Lozovskiy, M.Olshanskii, V.Salamatova, Yu.Vassilevski. An unconditionally stable semi-implicit FSI finite element method. *Comput.Methods Appl.Mech.Engrg.*, 297, 2015

Numerical tests

Numerical tests in 2D:

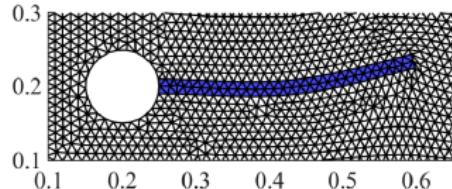
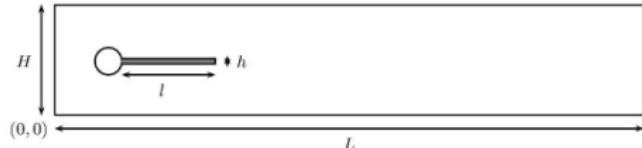
- ▶ FSI3: elastic beam in fluid (sketch)
- ▶ Blood vessel with aneurysm

Numerical test in 3D:

- ▶ Silicone filament in glycerol

Validation in 2D: FSI3 benchmark problem

S. Turek and J. Hron. Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow. In: *Fluid-structure interaction*, Springer Berlin Heidelberg, 371–385, 2006.

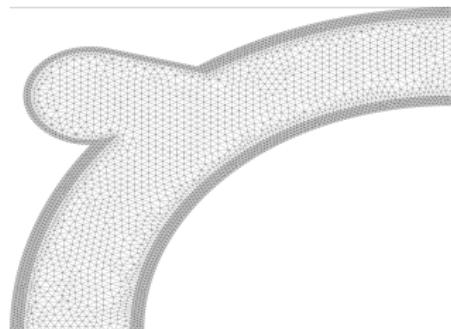


- ▶ fluid: 2D transient Navier-Stokes, $\rho_f = 1000$, $\mu_f = 1$
- ▶ stick: SVK constitutive relation, $\rho_s = 1000$, $\lambda_s = 4\mu_s = 8 \cdot 10^6$
- ▶ inflow: parabolic velocity profile
- ▶ outflow: “do-nothing”
- ▶ rigid walls: no-slip condition

Displacement extension in fluid domain: linear elasticity with $\mu_m = 20\mu_s$ and $\lambda_m = 20\lambda_s$ for adjacent to the beam elements

2D test: blood vessel with aneurysm

S. Turek et al. Numerical simulation and benchmarking of a monolithic multigrid solver for fluid-structure interaction problems with application to hemodynamics. In: *Fluid Structure Interaction II*, Springer Berlin Heidelberg, 193–220, 2010.



- ▶ Investigating sensitivity to compressibility of the vessel material: measuring wall shear stress (WSS) since it serves as a good indicator for the risk of aneurysm rupture
- ▶ Showing reliability of the semi-implicit scheme for hemodynamic applications

2D test: blood vessel with aneurysm

S. Turek et al. Numerical simulation and benchmarking of a monolithic multigrid solver for fluid-structure interaction problems with application to hemodynamics. In: *Fluid Structure Interaction II*, Springer Berlin Heidelberg, 193–220, 2010.

- ▶ Material properties:

ρ_s	μ_s	ρ_f	μ_f
$1.12 \cdot 10^3 \text{ kg/m}^3$	270000 Pa	$1.035 \cdot 10^3 \text{ kg/m}^3$	$3.4983 \cdot 10^{-3} \text{ Pa} \cdot \text{s}$

- ▶ Weakly compressible neo-Hookean model:

$$\boldsymbol{\sigma}_s = \frac{\mu_s}{J^2} \left(\mathbf{F} \mathbf{F}^T - \frac{1}{2} \text{tr}(\mathbf{F} \mathbf{F}^T) \mathbf{I} \right) + \left(\lambda_s + \frac{2\mu_s}{3} \right) (J-1) \mathbf{I}, \quad \lambda_s \rightarrow \infty$$

Extrapolation is used in the model to retain semi-implicitness

- ▶ Pulsatile parabolic inflow profile:

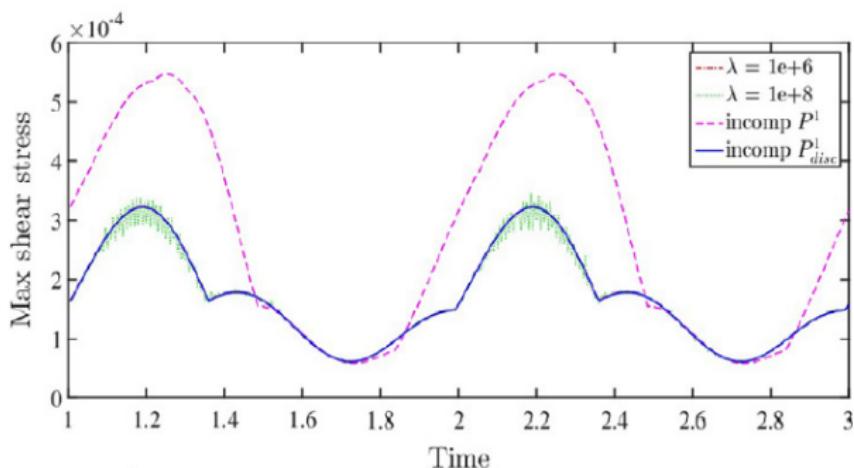
$$v_1(0, y, t) = -50(8-y)(y-6)(1 + 0.75 \sin(2\pi t)), \quad 6 \leq y \leq 8.$$

- ▶ λ_s takes values 10^4 , 10^6 , 10^8 kPa, i.e. Poisson's ratio $\nu \rightarrow 0.5$.
- ▶ Time step $\Delta t = 10^{-3}$ s until $T = 3$ s.
- ▶ Elasticity based displacement extension with $\mu_m = \mu_s$, $\lambda_m = 4\lambda_s$.

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WSS for weakly incompressible and fully incompressible cases, with continuous and discontinuous pressure at the interface:



Best choices (area of wall, WSS): Neo-Hookean compressible with moderate λ_s and incompressible with discontinuous pressures.

3D: silicone filament in glycerol

Benchmark challenge for CMBE 2015, Paris

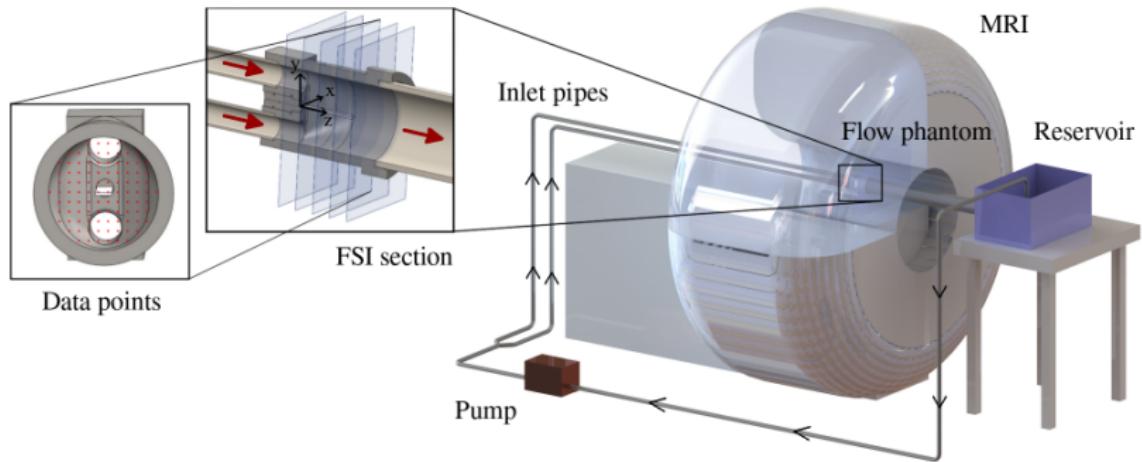
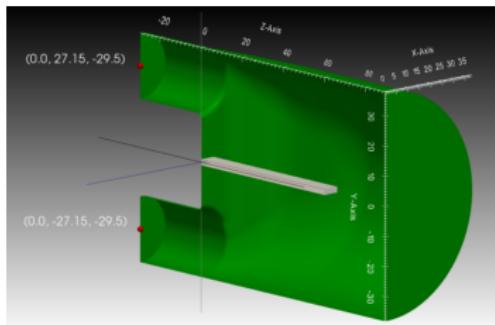
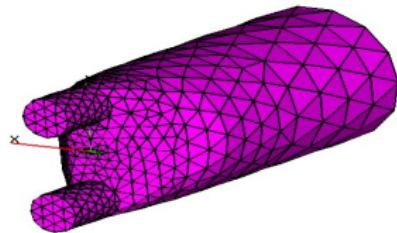
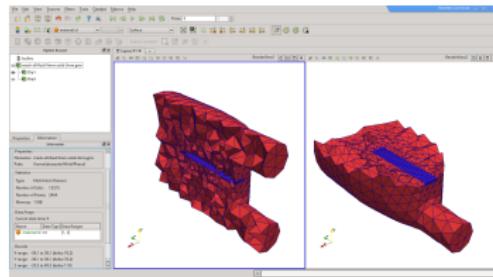


Image from A. Hessenthaler et al. Experiment for validation of fluid-structure interaction models and algorithms. *Int. J. for Numer. Meth. Biomed. Engng.*, 2016

3D: silicone filament in glycerol



Meshed volume: original and extended domains.



3D: silicone filament in glycerol

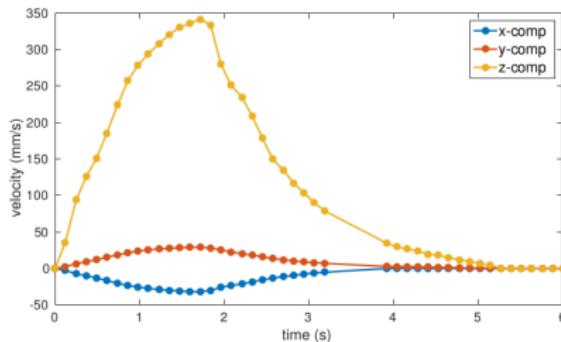
SVK material

M. Landajuela et al. Coupling schemes for the FSI forward prediction challenge: comparative study and validation. In: *Int. J. for Numer. Meth. in Biomed. Engng.*, 2016.

- ▶ $\rho_s = 1.063 \cdot 10^{-3} \text{ g mm}^{-3}$, $\lambda_s = 140.12 \text{ kg s}^{-2}\text{mm}^{-1}$, $\mu_s = 82.2 \text{ kg s}^{-2}\text{mm}^{-1}$, gravity **not** neglected!
- ▶ Two inflow regimes:

velocity	Phase I	Phase II
	stationary	pulsatile
ρ_f	$1.1633 \cdot 10^{-3} \text{ g mm}^3$	$1.164 \cdot 10^{-3} \text{ g mm}^{-3}$
μ_f	$12.5 \cdot 10^{-3} \text{ g mm}^{-1}\text{s}^{-1}$	$13.37 \cdot 10^{-3} \text{ g mm}^{-1}\text{s}^{-1}$

- ▶ Inflow velocities for one cycle of frequency 1/6 Hz for phase II:



3D: silicone filament in glycerol

SVK material

- ▶ Simulation was run with $\Delta t = 10^{-2}$ s
- ▶ The filament is lighter than the fluid and deflects upward
- ▶ Linear elasticity model is used for the **update** of the displacement extension in Ω_f ! The PDE model is nonlinear due to mapping to the reference domain. The Lame parameters are heterogeneous, i.e. element-volume dependent:

$$\lambda_m = 16\mu_m = 16 \frac{\mu_s}{V_e^{1.2}}$$

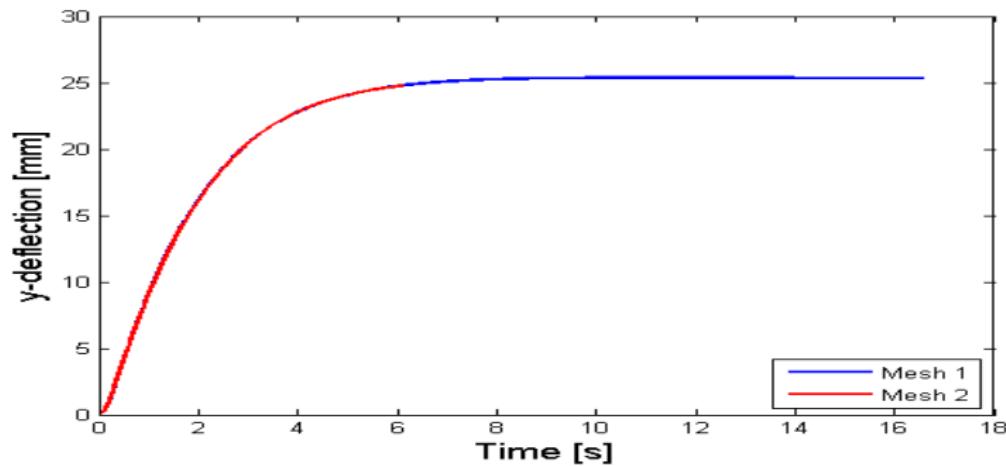
- ▶ Multi-frontal massively parallel sparse direct solver (MUMPS) to solve the linear system at every time step

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	# of cells in Ω_f	# of cells in Ω_s	# of DOFs
Mesh 1	28712	733	259914
Mesh 2	51496	733	459984

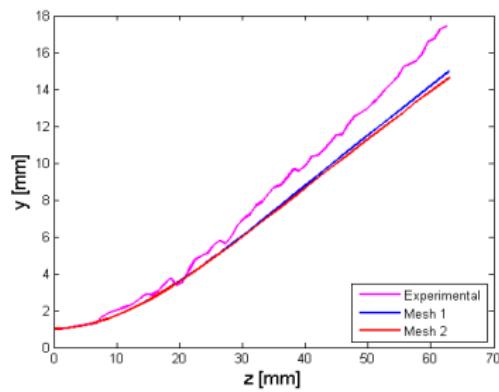
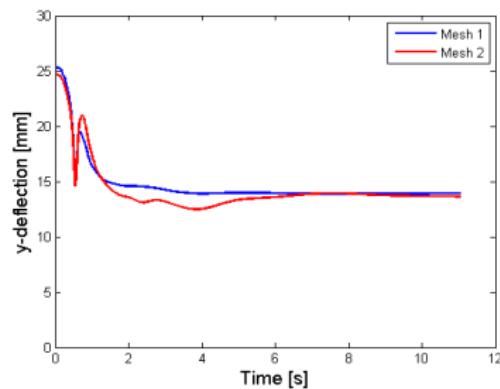
Deflection due to buoyancy force:



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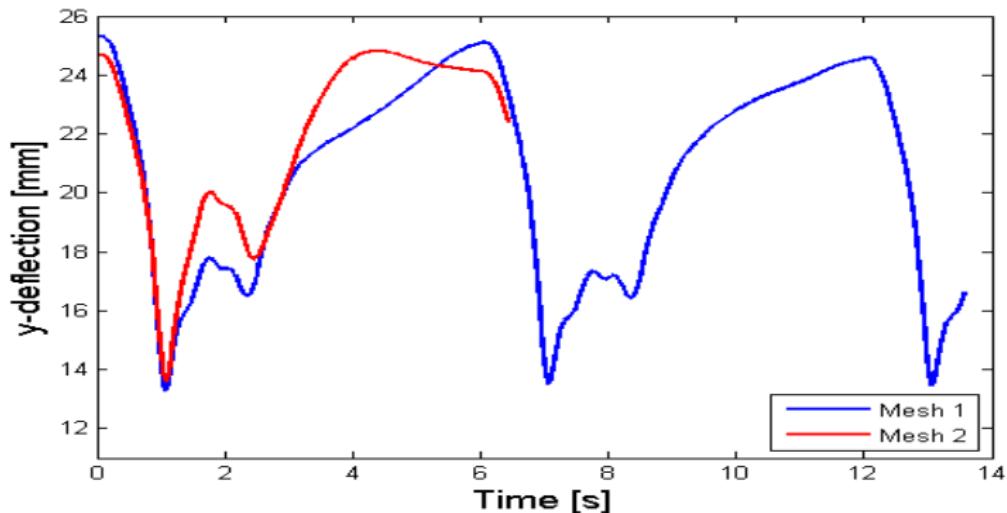
Phase I:



3D: silicone filament in glycerol

SVK material

Phase II:



Conclusions for Part I

- ▶ We proposed unconditionally stable semi-implicit ALE FE scheme for FSI
- ▶ Only one linear system is solved per time step
- ▶ The scheme can incorporate diverse elasticity models
- ▶ Works robustly in 2D and 3D and handles various time-discretizations
- ▶ Drawback: the scheme may suffer from mesh tangling for large deformations, and the cure is ad-hoc.

Incompressible fluid flow in a time-dependent domain

Navier-Stokes equations in a time-dependent domain

Prerequisites

- ▶ reference domain Ω_0

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- ▶ Cauchy stress tensor $\boldsymbol{\sigma}$
- ▶ pressure p
- ▶ density ρ is constant

Incompressible fluid flow in a moving domain

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Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = (J\rho_f)^{-1} \operatorname{div}(J\boldsymbol{\sigma}_f \mathbf{F}^{-T}) - \nabla \mathbf{v} \left(\mathbf{F}^{-1} \left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} \right) \right) \quad \text{in } \Omega_0$$

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$$\operatorname{div}(J\mathbf{F}^{-1}\mathbf{v}) = 0 \quad \text{in } \Omega_0 \quad \text{or} \quad J\nabla \mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_0$$

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Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f \mathbf{I} + \mu_f ((\nabla \mathbf{v}) \mathbf{F}^{-1} + \mathbf{F}^{-T} (\nabla \mathbf{v})^T) \quad \text{in } \Omega_0$$

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Mapping ξ does not define material trajectories \rightarrow quasi-Lagrangian formulation

Finite element scheme

Let $\mathbb{V}_h, \mathbb{Q}_h$ be Taylor-Hood P_2/P_1 finite element spaces.

Find $\{\mathbf{v}_h^k, p_h^k\} \in \mathbb{V}_h \times \mathbb{Q}_h$ satisfying b.c.

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$$\begin{aligned} & \int_{\Omega_0} J_k \frac{\mathbf{v}_h^k - \mathbf{v}_h^{k-1}}{\Delta t} \cdot \psi \, d\mathbf{x} + \int_{\Omega_0} J_k \nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \left(\mathbf{v}_h^{k-1} - \frac{\xi^k - \xi^{k-1}}{\Delta t} \right) \cdot \psi \, d\mathbf{x} - \\ & \int_{\Omega_0} J_k p_h^k \mathbf{F}_k^{-T} : \nabla \psi \, d\mathbf{x} + \int_{\Omega_0} J_k q \mathbf{F}_k^{-T} : \nabla \mathbf{v}_h^k \, d\mathbf{x} + \\ & \int_{\Omega_0} \nu J_k (\nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \mathbf{F}_k^{-T} + \mathbf{F}_k^{-T} (\nabla \mathbf{v}_h^k)^T \mathbf{F}_k^{-T}) : \nabla \psi \, d\mathbf{x} = 0 \\ & \int_{\Omega_0} J_k \nabla \mathbf{v}^{k+1} : \mathbf{F}^{-T} (\tilde{\mathbf{u}}^k) q \, d\Omega = 0 \end{aligned}$$

for all ψ and q from the appropriate FE spaces

Finite element scheme

The scheme

- ▶ semi-implicit
- ▶ produces one linear system per time step
- ▶ first order in time (may be generalized to the second order)

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- ▶ semi-implicit
- ▶ produces one linear system per time step
- ▶ first order in time (may be generalized to the second order)
- ▶ unconditionally stable (no CFL restriction) and 2nd order accurate, proved with assumptions:
 - ▶ $\inf_Q J \geq c_J > 0, \quad \sup_Q (\|\mathbf{F}\|_F + \|\mathbf{F}^{-1}\|_F) \leq C_F$
 - ▶ LBB-stable pairs (e.g. P_2/P_1)
 - ▶ Δt is not large

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling*, 32, 2017

A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A quasi-Lagrangian finite element method for the Navier-Stokes equations in a time-dependent domain. Submitted to *CMAME*

Stability estimate for the FE solution

Let $\partial\Omega(t) = \partial\Omega^{ns}(t)$ and ξ_t be given on $\partial\Omega^{ns}(t)$.

Then there exists $\mathbf{v}_1 \in C^1(Q)^d$, $\mathbf{v}_1 = \xi_t$, $\operatorname{div}(\mathcal{J}\mathbf{F}^{-1}\mathbf{v}_1) = 0$
[Miyakawa 1982]

and we can decompose the solution $\mathbf{v} = \mathbf{w} + \mathbf{v}_1$, $\mathbf{w} = 0$ on $\partial\Omega^{ns}$

Stability estimate for the FE solution

Stability estimate for \mathbf{w}_h^n FE approximation of \mathbf{w}^n :

$$C_1 \|\nabla \mathbf{v}_1^k\| \leq \nu/2:$$

$$\frac{1}{2} \|\mathbf{w}_h^n\|_n^2 + \nu \sum_{k=1}^n \Delta t \|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \leq \frac{1}{2} \|\mathbf{w}_0\|_0^2 + C \sum_{k=1}^n \Delta t \|\tilde{\mathbf{f}}^k\|^2$$

$$\mathbf{D}_k(\mathbf{v}) := \frac{1}{2} (\nabla \mathbf{v} \mathbf{F}_k^{-1} + \mathbf{F}_k^{-T} (\nabla \mathbf{v})^T)$$

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$$\frac{1}{2} \|\mathbf{w}_h^n\|_n^2 + \nu \sum_{k=1}^n \Delta t \|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \leq e^{\frac{2C_2}{\alpha} T} \left(\frac{1}{2} \|\mathbf{w}_0\|_0^2 + C \sum_{k=1}^n \Delta t \|\tilde{\mathbf{f}}^k\|^2 \right),$$

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$$\text{if } (1 - 2C_2 \Delta t) = \alpha > 0$$

3D: left ventricle of a human heart

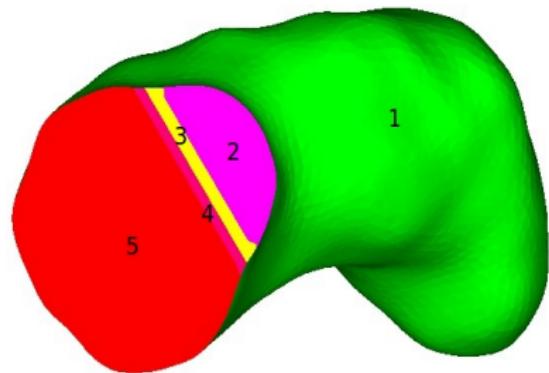


Figure: Left ventricle

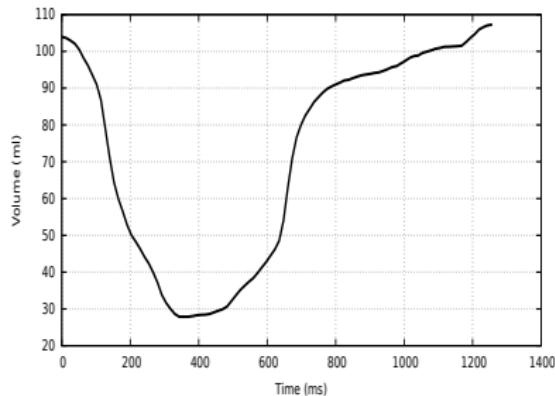
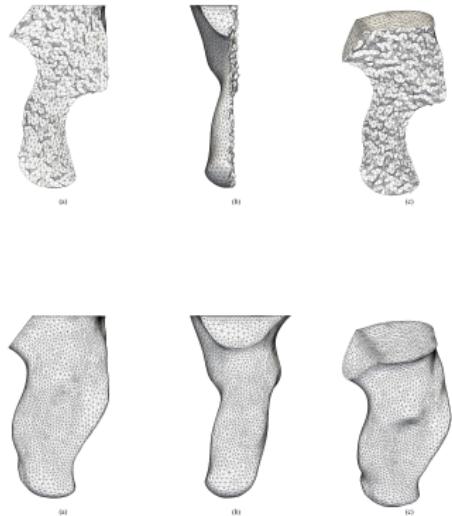


Figure: Ventricle volume

The law of motion for the ventricle walls is known thanks to ceCT scans → 100 mesh files with time gap 0.0127 s → \mathbf{u} given as input → FSI reduced to NSE in a moving domain

- ▶ 2 - aortic valve (outflow)
- ▶ 5 - mitral valve (inflow)

3D: left ventricle of a human heart



- ▶ Quasi-uniform mesh: 14033 vertices, 69257 elements, 88150 edges.
- ▶ Boundary conditions: Dirichlet
 $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ except:
 - ▶ Do-nothing on aortal valve during systole
 - ▶ Do-nothing on mitral valve during diastole
- ▶ Time step 0.0127 s is too large! \implies refined to $\Delta t = 0.0127/20$ s \implies Cubic-splined \mathbf{u} .
- ▶ Blood parameters: $\rho_f = 10^3$ kg/m³, $\mu_f = 4 \cdot 10^{-3}$ Pa · s.

3D: left ventricle of a human heart

DNS resulted in convective instability during sharp deformation phases.
We use a simple Smagorinsky dissipation model:

$$\mathbf{z}^{k-1} := \mathbf{v}^{k-1} - \frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\Delta t},$$

$$\begin{aligned} & \int_{\Omega(t^{k-1})} \frac{\mathbf{v}^k - \mathbf{v}^{k-1}}{\Delta t} \cdot \psi \, d\mathbf{x} + \int_{\Omega(t^{k-1})} \nabla \mathbf{v}^k \cdot \mathbf{z}^{k-1} \cdot \psi \, d\mathbf{x} \\ & - \int_{\Omega(t^{k-1})} s^k \operatorname{div} \psi \, d\mathbf{x} + \int_{\Omega(t^{k-1})} q \operatorname{div} \mathbf{v}^k \, d\mathbf{x} + \int_{\Omega(t^{k-1})} 2\nu \{\nabla \mathbf{v}^k\}_s : \nabla \psi \, d\mathbf{x} + \\ & \sum_e \int_{\Omega_e(t^{k-1})} 2\nu_T^{k-1} \{\nabla \mathbf{v}^k\}_s : \nabla \psi \, d\mathbf{x} = 0, \end{aligned}$$

where

$$\nu_T^{k-1} = 0.04 h_e^2 \sqrt{2 \{\nabla \mathbf{z}^{k-1}\}_s : \nabla \mathbf{z}^{k-1}}.$$

Worked for the entire cardiac cycle with the original viscosity and mesh!

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- ▶ The scheme is proven to be second order accurate in space
- ▶ Only one linear system is solved per time step
- ▶ The scheme was applied to blood flow simulation in a geometrical dynamic model of the left ventricle

