Reaction-diffusion equations with density dependent diffusion

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Application

1) Hemotaxis as chemical regulator of individual space activity

2) Sexual behavior in sparse populations

3) Genetic waves long time approximation

Basic equation

$$u_t = D(u)\nabla_x \left(N(u)\nabla_x u\right) + F(u)$$

Here $x \in \Omega \subset \mathbf{R}^{\mathrm{n}}$ – space variables, $t \in \mathbf{R}_+$ – time, $u = u(t, x)$ – phase variable

internal diffusion $N(u) > N_0 > 0$ external one $D(u) > D_0 > 0$.

Constant diffusion case

$$D(u) \equiv N(u) \equiv 1$$

Instability in bounded convex domain

If $\Omega = \mathrm{co}\,\Omega \subset\subset \mathbf{R}^{\mathrm{n}},\ D(u) \equiv 1$

$$(\nabla_x u, \nu)\,|_{\partial\Omega} = 0,\ \ \nu \perp \partial\Omega$$

Then any stationary nonconstant solution of the basic equation is not steady (e.g. in norm $C(\Omega)$).

Instability on real line

For $\Omega = \mathbf{R}$ the instability result is true if no one staying wave exists. E.g. it's true in the case

$$(u_1 < u_2) \,\&\, (F(u_1) = F(u_2) = 0) \Rightarrow$$
$$(J(u_1) \neq J(u_2))\,.$$

Here and later

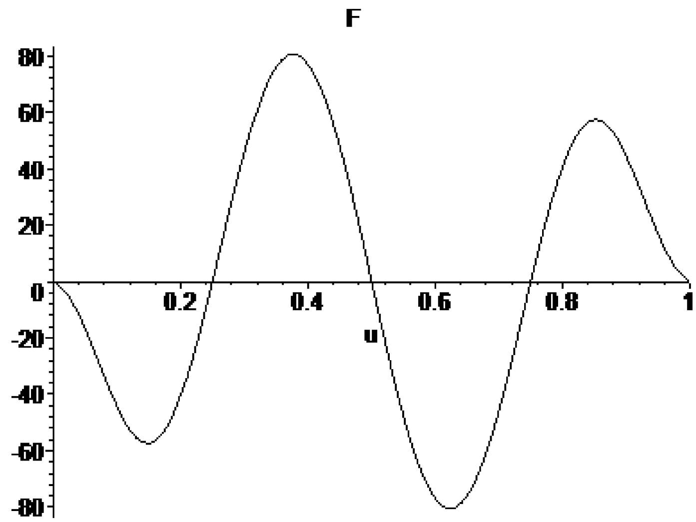
$$J(U) = \int_0^U \frac{F(u)N(u)}{D(u)} du$$

Stabilization to dominating equilibrium

Let $\Omega = \mathbf{R^n}$, $I = [0,1]$. $F(0) \geq 0$, $F(1) \leq 0$.

Stationary solution $\tilde{u}(x) \equiv U \in I$,(i.e. $F(U) = 0$) – is said to be *dominating* in I, iff $J(U) > J(u)$ for $u \in I, \quad u \neq U$. It's stable as $(u - U)F(u) < 0$ for $u \in O(U)$, $O(U) \subset I$

"Almost"global attractor for solutions with initials localized on I.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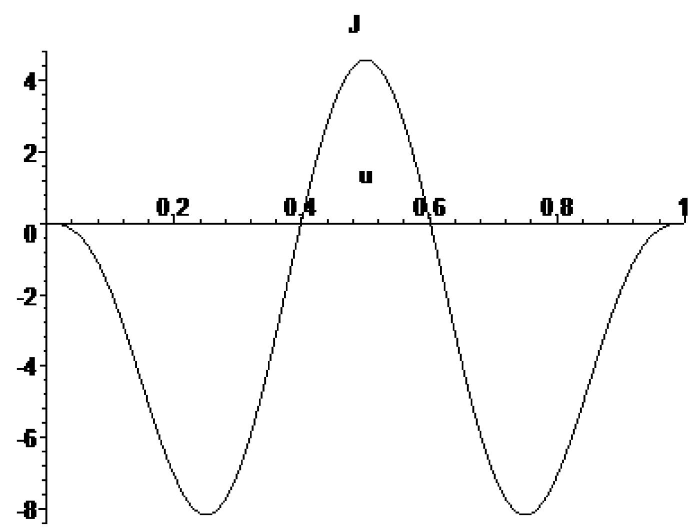
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Let $\tilde{u}(x) \equiv U$ – dominating equilibrium on I, and some neigborhood $O(U) \subset I$ don't include other equilibria. Then for any segment $[A, B] \subset O(U), A < U < B$ there exists $X > 0$ such that the solution $u(x,t)$ with initial $u(x,0) \in I$ and $u(x,0) \in [A, B]$ for $|x| < X$, stabilizes to U.

Single travelling wave

$$\Omega = \mathbf{R},\ F(0) = F(1) = 0$$

Wave solution (travelling wave)

$$u(x,t) = U(\xi),\ \xi = x + ct,\ U_\xi > 0$$

$$U(-\infty) = 0,\ U(+\infty) = 1, c \in \mathbf{R}$$

$$P(\xi) = N(U(\xi))U_\xi(\xi)$$

$$\begin{cases} U_\xi = \frac{P}{N(U)} \\ P_\xi = \frac{cP}{N(U)D(U)} - \frac{F(U)}{D(U)}. \end{cases}$$

$$\frac{dP}{dU} = \frac{c}{D(U)} - \frac{F(U)N(U)}{D(U)P}$$

$$J(1) = c\int_0^1 \frac{P(U)}{D(U)} dU \Rightarrow \operatorname{sign} J(1) = \operatorname{sign} c$$

Asymptotic solutions – ones, meeting the system and one of boundary conditions. Comparison technique for them results in existence and uniqueness theorems.

a) *Kolmogorov case*

$$F(u) > 0,\ u \in (0,1)$$

$$\forall c \geq c^*,\ c^* \in [c_m, c_M],$$

$$c_m = 2\sqrt{N\,(0)\,D\,(0)\,F_u\,(0)},$$

$$c_M = 2\sqrt{\sup_{u \in (U_1, U_2)} \left(N(u)F(u) \Big/ \int\limits_{U_1}^{u} \frac{dv}{D(v)} \right)}$$

b) *Trigger case*

$$F(\varepsilon) < 0,\ F(1-\varepsilon) > 0,\ \varepsilon > 0$$

$$(i)\&(ii) \Rightarrow \exists!\, c \geq 0$$

$$i)\,\forall u \in (0,1), \qquad J(u) < J(1),$$

$$ii)\bar{u} \in (0,1) \& F(\bar{u}) = 0 \Rightarrow J(\bar{u}) < 0$$

P.S. 1) In (ii) it's enought to check zeros $\bar{u}$ with:

$$F(\bar{u} - \varepsilon) > 0,\ \forall \varepsilon \in (0, \varepsilon_o), \varepsilon_o > 0$$

2) If $J(1) < 0$ the change

$\hat{c} = -c$, $\hat{x} = -x$, $\hat{u} = 1{-}u$, $\hat{F} = -F(1{-}\hat{u})$,

$\hat{D}(\hat{u}) = D(1 - \hat{u})$, $\hat{N}(\hat{u}) = N(1 - \hat{u})$,

Trigger wave chains

Zero $\bar{u}$ is stable (with respect to ODE $\frac{du}{dt} = F(u)$) iff

$$\pm F(\bar{u} \pm \varepsilon) \leq 0, \; \forall \varepsilon \in (0, \varepsilon_o), \varepsilon_o > 0$$

Let $0 = u_0 < u_1 < \ldots < u_k = 1$ – isolated zeros, u_i, $i \geq 1$ – stable

$$i < j, \; c_{ij} \in \mathbf{R} :$$

$$U(-\infty) = u_i, \; U(+\infty) = u_j$$

If for $0 \leq q < m < l \leq k$ $\exists \, c_{qm} < c_{ml}$. Then $\exists c_{ql} \in \big(c_{qm}, c_{ml}\big)$.

Minimal chains

For any full set of zeros

$$\{u_i\}, \; u_1 < u_2 < \ldots < u_k$$

$$i_j \in \{1, \ldots, k\} : \exists! \, \{c_{i_j i_{j+1}} \geq c_{i_{j+1} i_{j+2}}\}$$

Kolmogorov wave chains

u_0 – unstable $\exists c_{0j} \in I_j$, $j \geq 2$ where by induction $I_j = [c_K^j, \hat{c}^j)$, with $\hat{c}^j$ – minimal velocity in trigger chain for $\{u_p\}$. $p = 1, \ldots, j$, and $c_K^j \in I_{j'}$, $j' < j$.

Stabilization to wave chains

$$F(0) = F(1) = 0,$$

$$u_x(x,0) \geq 0 \Rightarrow u_x(x,t) > 0$$

for $Q(u,t) = u_x(u,t)$

$$Q_t = Q^2 \left(D\,(QN)_u + \frac{F}{Q} \right)_u =$$
$$= Q^2 (D(QN)_u)_u + F_u Q - F Q_u$$

$$\pm \frac{\alpha_\pm(t)}{dt} = \pm F(\alpha_\pm) \geq 0$$

$$\exists! Q(u,t),\ Q(\pm\alpha_\pm(t),t)=0$$

$$\int\limits_{\alpha_\pm(t)}^{\alpha_\pm(t)\mp\varepsilon} \frac{du}{Q(u,t)} = \infty,\ t>0$$

$$Q(u,0)\geq 0,\ u\in[0,1]$$
$$\&\neq 0,\ u\in(\alpha_-(0),\ \alpha_+(0))$$

Sub- & supersolutions

$(Q_t =) \rightarrow$

$$\begin{cases} (Q_t \leq)\text{– regular subsolution (RDS)} \\ (Q_t \geq)\text{– regular supersolution (RUS)} \end{cases}$$

Maximum of RDS is a subsolution (DS).
Minimum of RUS is a supersolution (US).

1) solution is both RDS and RUS

2) US $\geq$ DS if at $t=0$

3) If $Q_+(u)\geq 0$ – stationary US (SUS) and not a solution and $Q(u,t)$ with $Q(u,0)=$

$Q_+(u)$ – solution such that on boundaries $\frac{d}{dt}Q(0,t) \leq 0$, $\frac{d}{dt}Q(1,t) \leq 0$,

then $\frac{\partial}{\partial t}Q(u,t) \leq 0$, $\neq 0$ if $Q(u,t) \neq 0$. So, $Q(u,\bar{t})$ for $\bar{t} > 0$ is SUS.

4) If $Q_-(u) \leq Q(u,\bar{t}) \leq Q_+(u)$ at some $\bar{t} \geq 0$, and $\exists! \bar{Q}(u)$ – SS, such that

$$Q_-(u) \leq \bar{Q}(u) \leq Q_+(u),$$

then $Q(u,t) \to \bar{Q}(u)$, $t \to \infty$.

Convergence on phase plane results in convergence to shifted wave with proper velocity.

Supersolutions in single wave case

In case $F(0) = F(1) = 0$ let

$$M > \sup_u \max\{Q(u,0), Q_i(u)\},$$

with $Q_i(u)$ – all possible travelling wave solutions on $(0,1)$,

$$\hat{Q}(u) = M + \int_0^u \frac{1-v}{D(v)} dv,$$

$$A^2 \geq \sup_u \left\{ \left| \left(\frac{F(u)N(u)}{\hat{Q}(u)} \right)_u \right| \right\}$$

$\bar{Q}(u) = A\frac{\hat{Q}(u)}{N(u)}$ – SUS, so that the solution $Q_M(u,t)$ with $Q_M(u,0) = \bar{Q}(u)$ and

$$Q_M(0,t) = Q_M(1,t) = 0,$$

for $t > 0$, is nonincreasing

Subsolutions in single wave case

1) Basic DS.

On connected sets $I_\pm \subset (0,1)$, where $\pm F(u) > 0$.

a) $\omega \in I_-$.

$Q_-(u) = \max\{0, \varepsilon F(u)(\omega - u)\}$, $\varepsilon > 0$.

b) $\omega \in I_+$.

$Q_-(u) = \max\{0, \Phi(u)\}$ with rather small asymptotic solution $\Phi(u)$, leaving unstable zero.

Bridges over unstable zero $\hat{u}$

$$Q_-(u, \hat{u}, \varepsilon) = \frac{\sqrt{2\,(J(\hat{u}) + \varepsilon - J(u))}}{N(u)}$$

Bridges over stable zero $\hat{u}$

$\hat{u} \in (0,1)$ – single inner stable zero. $F'(\bar{u}) < 0$.

1) $c_1 \geq c_2$ – combinatorics of single waves and zeros.

2) Let $c_1 < c_2$, with the first wave entering an equilibrium $\hat{u}$, and the second one leaving it. Then $Q(\hat{u}, 0) > 0$ results in $Q(\hat{u}, t) > \varepsilon > 0$, so under ε one can construct the bridge – wave solution with velocity $c \in (c_1, c_2)$

Super- and subsolutions in Kolmogorov case $F_u(0) > 0$

from characteristic equation

$$c = \tilde{c}(q) = qD(0)N(0) + \frac{F_u(0)}{q},$$

with $\tilde{q}(c) = \min\{q > 0 : \ \tilde{c}(q) = c\}$

Function $c = \tilde{c}(q) > 0$ has minimum c_m at $\hat{q} = \sqrt{\frac{F_u(0)}{(D(0)N(0))}}$

1) In the case with minimal velocity c^* one can get $\Phi(u) = \max \Phi_\lambda(u)$ with $\Phi_\lambda(u) \leq Q(u, 0)$ – trajectories, leaving unstable zero with $c \to c^* + 0$.

2) Real wave velocity c is defined by initial asymptotic at $u = 0$.

E.g. if $\exists q = \frac{dQ(u,0)}{du}|_{u=0} \in [0, q^*]$, for

$q^* \leq \hat{q}$ such that $c^* = \tilde{c}(q^*) \geq c_m$ then $c = \tilde{c}(q) \geq c^*$. Otherwise $(q > q^*)$ one has $c = c^*$.

DS are constructed as above. US for proper value of q is constructed at later time moment.

The case of minimal velocity

Theorem Let nonnegative continuous function $Q(u, 0)$ is positive over $(0, \beta)$, with $F(\beta) > 0$ or $\beta \in \{u_i\}, \quad i = 1, \ldots, k$. Then under the inequality

$$\lim \inf_{u \rightarrow +0} \frac{Q(u, 0)}{u} \geq \tilde{q}(c^*)$$

the true solution $Q(u, t)$ converges on phase plain for $t \rightarrow +\infty$ to Kolmogorov's minimal true chain on interval $(0, u_+)$, where $u_+ = \min u_i \geq \beta, \; i = 1, \ldots, k,$ with minimal first wave velocity c^{j_M}.

Conclusion. In the case of strictly increasing over x initial $u(x,0)$, vanishing at some finite $\bar{x}$ with $u_x(\bar{x},0) > 0$, or else with $u_x(x,0) > 0$ and $u_{xx}(x,0) \geq -\delta$ at $x \in (\bar{x}, \bar{x}+\varepsilon)$ for some $\varepsilon, \delta > 0$, the first wave velocity would be minimal, i.e. equal c^{j_M}.

The case of non-minimal velocity

For $c > c^*$ let $C_c(u)$ be the wave with velocity c.

Theorem. Let $\tilde{Q}(u,0) = u(q + g(u))$ with $q \in (0, \tilde{q}(c^*))$ and continuous over $[0,1]$ function $g(u) = o(1)$, for which the integral $\int\limits_{+0}^{1} \frac{g(u)}{u} du$ converges.

Then the true solution $\tilde{Q}(u,t)$ converges to the wave $C_{\tilde{c}(q)}(u)$ for $t \to +\infty$ in the following sence.

There exist functions $Q_{\pm}(u,t)$ such that $0 \leq Q_{-}(u,t) \leq Q(u,t) \leq Q_{+}(u,t)$, whereas $Q_{-}(u,t) \leq C_{\tilde{c}(q)}(u)$ converges to $C_{\tilde{c}(q)}(u)$ uniformly on compacts $I \subset (0,1)$, and for $Q_{+}(u,t) \geq C_{\tilde{c}(q)}(u)$ in the metric $\rho^t(Q_1, Q_2) =$

$$\int_t^{+\infty} \int_0^1 \frac{F(u)}{D(u)} \left| \frac{1}{Q_1(u,\tau)} - \frac{1}{Q_2(u,\tau)} \right| du d\tau$$

$$\rho^t \left(Q_{+}(u,t), C_{\tilde{c}(q)}(u) \right) \to 0$$

convergence takes plase.

The leader selection in a diffusion model of compeating species.

$x \in \mathbf{R}$, $i = 1, \ldots, N$, $u^i(x,t) \geq 0$. $D^i > 0$, $F^i(u) = u^i \left(M^i - \sum_{j=1}^{N} \gamma_{ij} u^j \right)$, $u = (u^1, \ldots, u^N)$ – population dencities, $M = (M^1, \ldots, M^N) > 0$ – maltusian parameters, $\Gamma = \|\gamma_{ij}\|$ – competition matrix, $\gamma_{ii} > 0$.

Reaction-diffusion system:

$$u^i_t = D^i u^i_{xx} + u^i \left(M^i - \sum_{j=1}^{N} \gamma_{ij} u^j \right)$$

$u^i(x,0) \geq 0$, The support $S^i = \operatorname{supp} u^i(x,0) = \operatorname{cl}\{x : u^i(x,0) > 0\} \neq \emptyset$.

$$S = \operatorname{co}\left(\bigcup_{i=1}^{N} S^i\right)$$ – *seat* is bounded.

(H1) The common ecological niche hypothesis: $\gamma_{ij} = \alpha_i \beta_j$,& $m^i = \frac{M^i}{\alpha_i}$ are different.

For $N = 1$ $c^1 = 2\sqrt{D^1 M^1}$, $\hat{u}^1 = \frac{M^1}{\gamma_{11}}$.

The spesies $i_1 \in \{1, \ldots, N\}$ is a *leader*, $\forall X > 0$ $\exists \hat{x} > X$: $\exists t_j > 0$:

1) $u^{i_1}(\hat{x}, t_j) > \delta^i$;

2) $u^k(x, t_j) < \delta^k$ $\forall k \neq i_1$ and $x \geq \hat{x}$.

(H2) Solution of the problem $i_1 \in \{1, \ldots, N\}$,

$$\sqrt{D^{i_1} M^{i_1}} = \max_i \left\{\sqrt{D^i M^i}\right\}.$$

is unique.

Theorem (H1), (H2) $\Rightarrow$ The leader exists and its number does not depend on initial finite distributions

Example: Diffusion model of genetic waves

Hypothesises

1. The Population is distributed in space with one variable $x \in \mathbf{R}$.

2. Phenotypic particularities differ due to one two-allelic gene with alleles A and a.

3. Total number of A and a alleles of fertile individuals is constant

$p(t, x) + q(t, x) = 1.$

4. Changes of alleles A and a number occurs through their carriers with genotypes AA, Aa, and aa having fitness (the alive factors to maturation age, i.e. from conception to fertile stage) α, β, γ. The departure rate from fertile stage (death-rate + aging) is constant.

5. Locally over x the hypothesis about full panmixia is fulfilled.

6. Crossbreeding is realized via the gametes spatial carrying, and not depends on genotype.

Integral model

The long-range genetic action operator at maturation time $h = 1$

$$K(p)(x,t) = \int_{-\infty}^{+\infty} k(x.\xi)p(\xi,t)d\xi$$

$$k(x.\xi) = \frac{e^{-\frac{(x-\xi)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}, \qquad K(1) = 1$$

(3) If $\varphi = K(p), \quad \psi = K(q) = 1 - \varphi$, $u_A = p\varphi$, $u_{Aa} = p\psi + q\varphi$, $u_a = q\psi$, that

$$\begin{cases} p_t = 2\alpha u_A + \beta u_{Aa} - pr, \\ q_t = \beta u_{Aa} + 2\gamma u_a - qr, \end{cases}$$

where $r : p + q = 1$ so $r = r(p + q) = 2(\alpha u_A + \beta u_{Aa} + \gamma u_a)$.

$$p_t = \varphi R(p) + [\beta p(q - p) - 2pq\gamma],$$

$$R(p) = 2pq(\alpha + \gamma) + \beta(q - p)^2 > 0$$

PDE aproximation

Decomposition

$$p(\xi,t) = p(x,t) + \frac{\partial p(x,t)}{\partial x}(\xi - x) +$$

$$\frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}(\xi - x)^2 + \dots$$

gives for small $\sigma > 0$

$$K(p)(x,t) = p(x,t) + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2} + \dots,$$

and equation

$$p_t = D(p)p_{xx} + F(p),$$

where $D(p) = \frac{\sigma^2}{2}R(p)$,

$$F(p) = pR(p) + [\beta p(q-p) - 2pq\gamma] = 2pq[\alpha p - \gamma q + \beta(q-p)].$$

Main results

Three equilibria:

$$p_0 = 0, \quad p_1 = 1, \quad p^* = \frac{\gamma - \beta}{\alpha + \gamma - 2\beta}.$$

$J = \int_0^1 \frac{F(u)}{D(u)} du$, sign $J =$ sign $\left(\frac{1}{2} - p^*\right)$

$\delta = \frac{\alpha+\gamma}{2}$. $\alpha > \gamma$

Intervals for β:

$$(-\infty, \gamma), \quad (\gamma, \delta), \quad (\delta, \alpha), \quad (\alpha, +\infty)$$

$R(p)$ concave $\delta > \beta$, convex $\delta < \beta$.

1. $\beta < \gamma$.

$F'(0) = 2(\beta - \gamma) < 0, \quad F'(1) = 2(\beta - \alpha) < 0$, $p^* \in \left(0, \frac{1}{2}\right)$, $J > 0$

The Wave solution = genetic wave of spreading of more strong allele A.

2. $\gamma < \beta < \delta$.

$F'(0) > 0, \quad F'(1) < 0$, and $p^* < 0$ so $F(p) > 0$ under $p \in (0,1)$. Wave solutions with velocity $c \geq c^* \geq c_K$, where c^* – minimum, but

$$c_K = 2\sqrt{D(0)F'(0)} = 2\sigma\sqrt{\beta(\beta - \gamma)}$$

– Kolmogorov's velocity.

3. $\delta < \beta < \alpha$.

$F'(0) > 0, \quad F'(1) < 0$, and $p^* > 1$, once again $F(p) > 0$ under $p \in (0,1)$.

Since function $D(p)$ is convex (maximum on the end of the interval $(0,1)$), but

$$F'(p) > 0 \Rightarrow F''(p) < 0$$

, that $C^* = C_K$

4. $\beta > \alpha$.

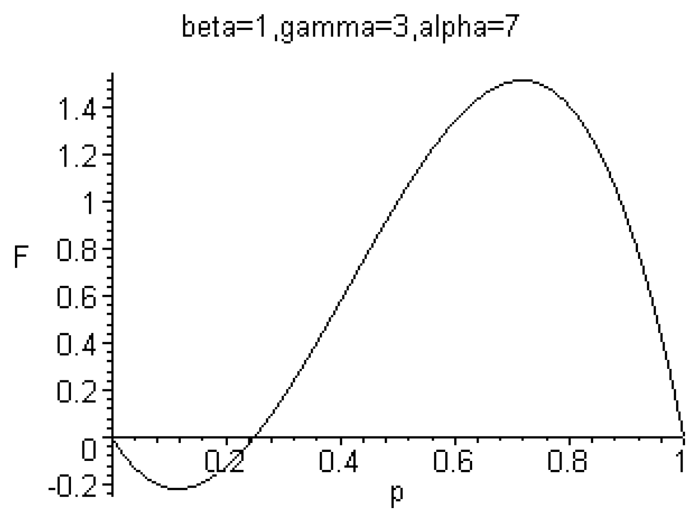
$F'(0) > 0, \quad F'(1) > 0$. Chain of two waves, scatterring from $p^* \in \left(\frac{1}{2}, 1\right)$. The velocities value spread from

$$c_K^0 = 2\sqrt{D(0)F'(0)} = 2\sigma\sqrt{\beta(\beta-\gamma)}$$

and

$$c_K^1 = 2\sqrt{D(1)F'(1)} = 2\sigma\sqrt{\beta(\beta-\alpha)}$$

to $+\infty$. Minimal value of the velocity follows from implication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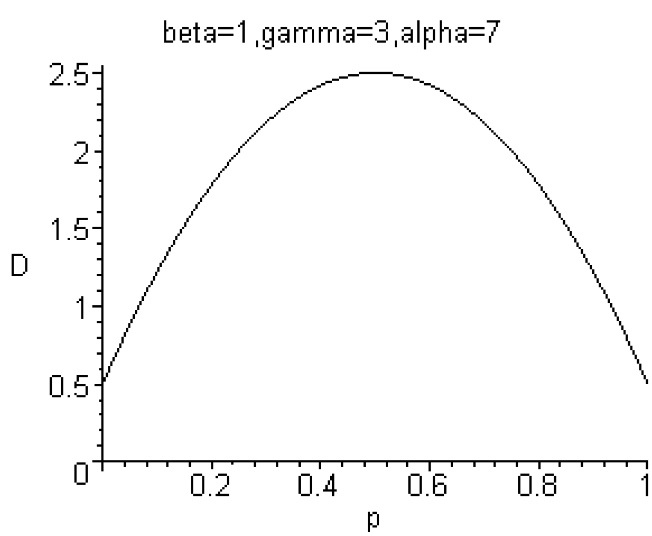
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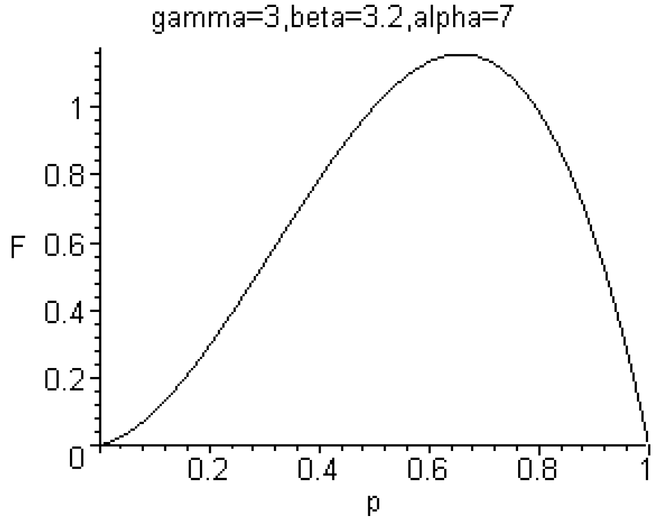
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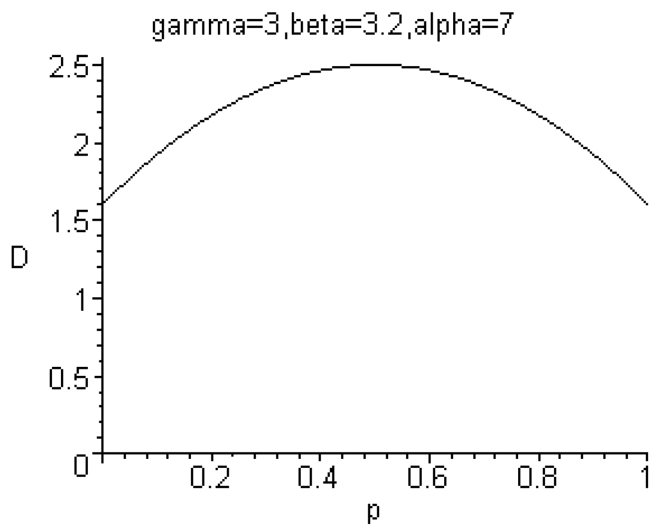
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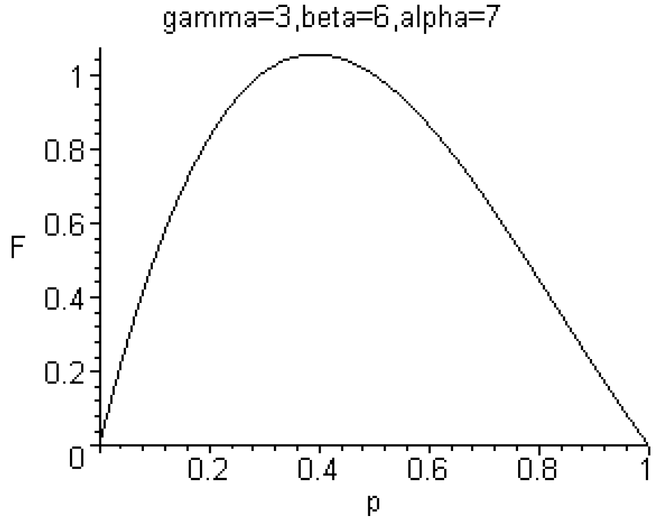

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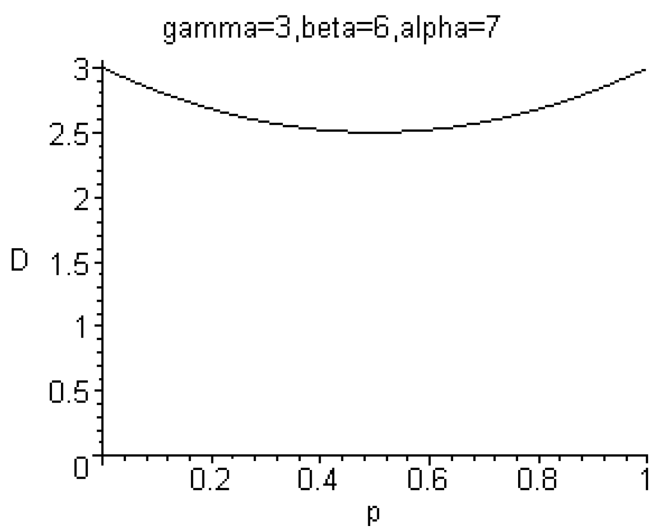

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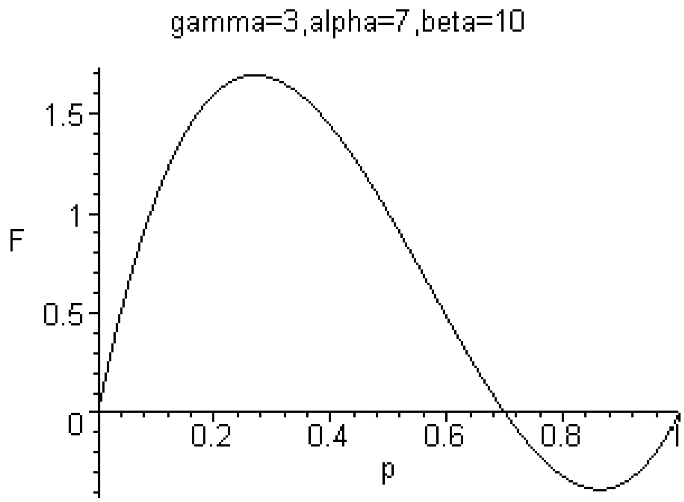
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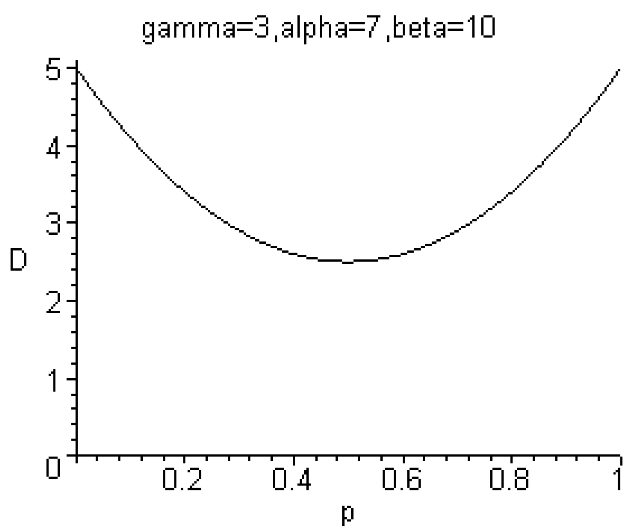
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