Lévy walks and anomalous transport on scale-free networks

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Lévy flight and Lévy walk

Observation for intracellular transport, cell movement, diffusion of tracers in fluid flows, animal foraging, human travel: path lengths ℓ have distribution

$$P(\ell) \sim 1/\ell^{1+\alpha}, \quad 0 < \alpha < 2$$

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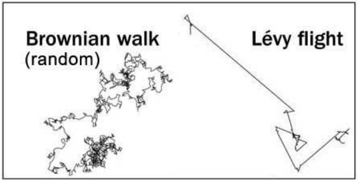
Lévy flight and Lévy walk are generalized random walk in which the step lengths during the walk are described by a "heavy-tailed" probability distribution.



What is the difference between Lévy flight and Lévy walk?

Lévy Flight vs. Lévy Walk

- Lévy Flight involves an instantaneous jump to a new position. It is a Markovian process!

  Fractional equation for walker's probability density ρ

  $$\frac{\partial \rho}{\partial t} = -D_{\alpha} \left(-\Delta\right)^{\frac{\alpha}{2}} \rho, \qquad x \in \mathbb{R}^2$$

  where $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian.

- Lévy Walk involves a random movement with finite velocity $\vec{v}$ constrained to ballistic cone $|\vec{x}| = |\vec{v}|t$.

  Because of non-Markovian nature of Lévy walk, it cannot be treated simply with a fractional Laplace operator.

  What is the governing integro-differential equation for the PDF ρ?

Approaches to Lévy Walks in 1D

CTRW Approach

Zaburdaev, Denisov, Klafter, Rev. Mod. Phys. 87, 483 (2015)

CTRW involves the joint probability density function (PDF) $\psi(x,\tau)$ of the running times and lengths of displacement

$$\psi(x,\tau) \sim \frac{1}{2}\delta(|x| - v\tau)\psi(\tau).$$

where $\psi(\tau)$ is the a "heavy-tailed" probability density for the running time

Structural Density Approach

Fedotov, Phys. Rev. E. 93, 020101 (2016)

1. A walker moves with finite velocity $\pm v$ for a random running time
2. With a rate $\lambda(\tau)$ the 'run' ends and a new direction is chosen.

Define a structural density $n_{\pm}(x,t,\tau)$ of walkers at position x at time t with running time τ moving to the left $(-)$ or right $(+)$:

$$\frac{\partial n_{\pm}}{\partial t} + \frac{\partial n_{\pm}}{\partial \tau} \pm v\frac{\partial n_{\pm}}{\partial x} = -\lambda(\tau)n_{\pm}(x,t,\tau). \quad (1)$$

Advantages of the Mesoscopic Approach

Lévy walkers are persistent with the running rate

$$\lambda(\tau) = \frac{\mu}{\tau_0 + \tau},$$

where $\tau_0 > 0$ and $1 < \mu < 2$.

Why consider turning rates $\lambda(\tau)$ instead of the joint PDF $\psi(x, \tau)$ for running times and lengths of displacement?

- By explicitly introducing the running time, the motion becomes Markovian.
- For proliferating random walkers, reaction rates can be systematically included.
- Rates can be non-linear: $\lambda(\rho(x,t),\tau) = \frac{\mu}{\tau_0+\tau} + f[\rho(x,t)]$.
- Non-local behaviour with the interactions with other walkers can be included.

There is no consistent methodology for including these effects if starting from the PDF $\psi(x, \tau)$.

Single integro-differential wave equation for Lévy walk

If the running rate λ is constant, we obtain the telegraph equation

$$\frac{\partial^2 \rho}{\partial t^2} - v^2 \frac{\partial^2 \rho}{\partial x^2} + 2\lambda \frac{\partial \rho}{\partial t} = 0$$

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For the general running time density $\psi(\tau)$, we obtain an integro-differential wave equation for the classical one-dimensional Lévy walk:

$$\frac{\partial^2 \rho}{\partial t^2} - v^2 \frac{\partial^2 \rho}{\partial x^2} + \int_0^t \int_V K(\tau) \varphi(u) \left(\frac{\partial}{\partial t} - u \frac{\partial}{\partial x} \right) \rho(x - u\tau, t - \tau)\, du d\tau = 0,$$

where v is a constant speed of walker, $\varphi(u)$ is the velocity jump density:

$$\varphi(u) = \frac{1}{2}\delta(u - v) + \frac{1}{2}\delta(u + v) \tag{2}$$

in the velocity space V. The standard memory kernel $K(\tau)$ is determined by its Laplace transform $\hat{K}(s) = \hat{\psi}(s)/\hat{\Psi}(s)$, where $\hat{\psi}(s)$ and $\hat{\Psi}(s)$ are the Laplace transforms of the running time density $\psi(\tau)$ and the survival function $\Psi(\tau)$ Fedotov, Phys. Rev. E. 93, 020101 (R) (2016)

Lévy Walks Emerge From Walker Alignment

The walkers can interact non-locally with each other, such that the running rate

$$\lambda_{\pm}(\tau, \rho_-, \rho_+) = \frac{\mu_{\pm}(\rho_-, \rho_+)}{\tau + \tau_0},$$

where $\rho_-(x,t)$ and $\rho_+(x,t)$ are the density of particles moving left and right,

$$\mu_{\pm}(\rho_-, \rho_+) = \mu \exp\left(-a \int_{\mathbb{R}} e^{-\frac{|z|}{l_a}} \left[\rho_{\pm}(x+z,t) - \rho_{\mp}(x+z,t)\right] dz\right) \quad (3)$$

expresses the alignment of walkers with interaction strength a, and characteristic length l_a.

Persistent walkers which align with their neighbors produce superdiffusive Lévy walks.

Fedotov and Korabel, Phys. Rev. E **95**, 030107(R) (2017).

Transport in Metapopulation Networks

Scale-Free Network: The power-law probability that a given node has k links (order k) to other nodes: $P(k) \sim k^{-\gamma}, \ \gamma \in [2, 3]$.

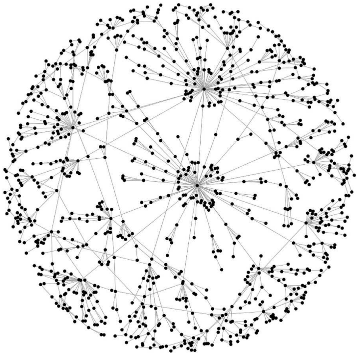

Figure : Barabási-Albert network,

(∗) Colizza and Vespignani, Phys. Rev. Lett. **99**, 148701 (2007).

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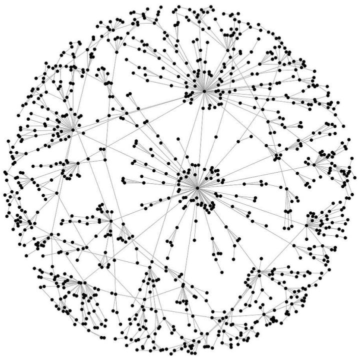

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Mean field transport equation:

$$\frac{dN_k(t)}{dt} = -\mathbb{I}_k(t) + k\sum_{k'} P(k'|k)\frac{\mathbb{I}_{k'}(t)}{k'}$$

N_k : mean number of individuals in node of order k;

$\mathbb{I}_k$: mean flux out of node of order k;

$P(k'|k)$: the probability of a link between nodes of order $k \to k'$. For $\mathbb{I}_k(t) = \lambda N_k(t)$:

$$N_k^{st} = k\frac{\langle N\rangle}{\langle k\rangle}$$

– well-connected nodes are more populous.(∗)

Human activity is not Poissonian!(†)

(†)A.-L. Barabási,

Axiom of Cumulative Inertia in Network Theory

Axiom of Cumulative Inertia:

An individual's escape probability from a node decreases with the (residence) time T spent in the node.

This is an empirical sociological law. The escape rate γ_k decreases with residence time

$$\gamma_k(\tau) = \frac{\mu_k}{\tau + \tau_0}, \quad \mu_k, \tau_0 > 0$$

Probability density function (PDF) of a residence time is

$$\psi_k(\tau) = \frac{\mu_k}{\tau + \tau_0} \left(\frac{\tau_0}{\tau + \tau_0} \right)^{\mu_k} \sim 1/\tau^{1+\mu_k},$$

Fedotov and Stage, Phys. Rev. Lett. **118**, 9 (2017).

How Does the Axiom of Cumulative Inertia Affect the Flux?

Non-Markovian behavior of individuals performing random walk on network occurs when individuals are trapped during the random time with non-exponential distribution.

Instead of the classical escape flux from the node with k links

$$\mathbb{I}_k = \lambda N_k(t),$$

the Axiom of Cumulative Inertia leads to a flux

$$\mathbb{I}_k = \frac{1}{\Gamma(1-\mu_k)\tau_0^{\mu_k}} \mathcal{D}_t^{1-\mu_k} N_k(t), \quad 0 < \mu_k < 1,$$

where the Riemann-Liouville (fractional) derivative $\mathcal{D}_t^{1-\mu}$ is defined as

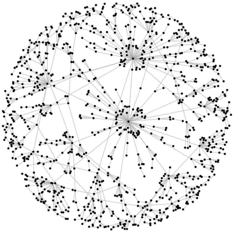
$$\mathcal{D}_t^{1-\mu_k} N_k(t) = \frac{1}{\Gamma(\mu_k)} \frac{\partial}{\partial t} \int_0^t \frac{N_k(u)du}{(t-u)^{1-\mu_k}} \tag{4}$$

How Does the Axiom of Cumulative Inertia Affect particle aggregation?

We consider the network for which only one node with, say, 2 links has anomalous behaviour: power law waiting time density

$$\phi(\tau) \sim \frac{1}{\tau^{1+\mu_2}}$$

with $0 < \mu_2 < 1$. The node's mean residence time $\langle T \rangle = \infty$.



Main result: ultimately all individuals are attracted to this anomalous node with $\mu_2 < 1$.

Anomalous Aggregation

What is happening inside the trapping (anomalous) node?

Consider the structural density of individuals at time t with residence time τ, $n_{trap}(t, \tau)$.

$$n_{trap}\left(t, \tau\right) \to \frac{N}{\Gamma(1-\mu_k)\Gamma(\mu_k)\tau^{\mu_k}\left(t-\tau\right)^{1-\mu_k}},$$

N is the total number of individuals in the network.

$\rightarrow$ Most individuals have been there for a long time, or are new arrivals.

Is this realistic?

Yes! Data: American MidWest

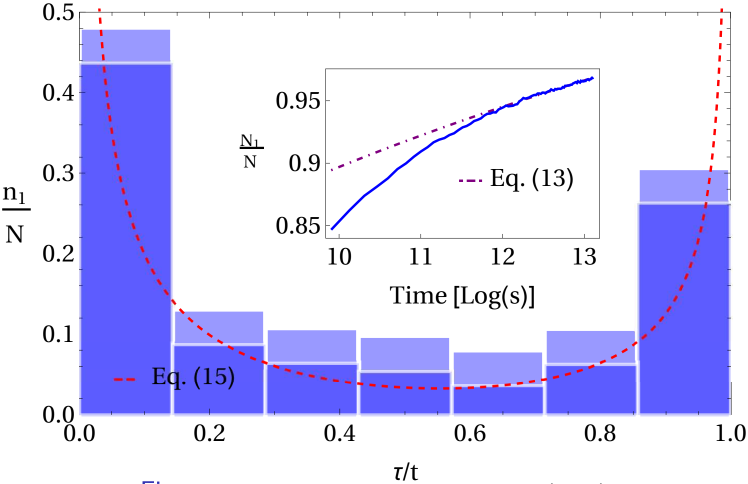


Figure : Fedotov and Stage, PRL **118**, 9 (2017)

Conclusions

• The mesoscopic description of non-Markovian reaction-transport processes on the network is still an open problem.